INTERNAL FORCES
Beams, Frames, Arches



Dr Ghayath Hallak

APPLICATIONS



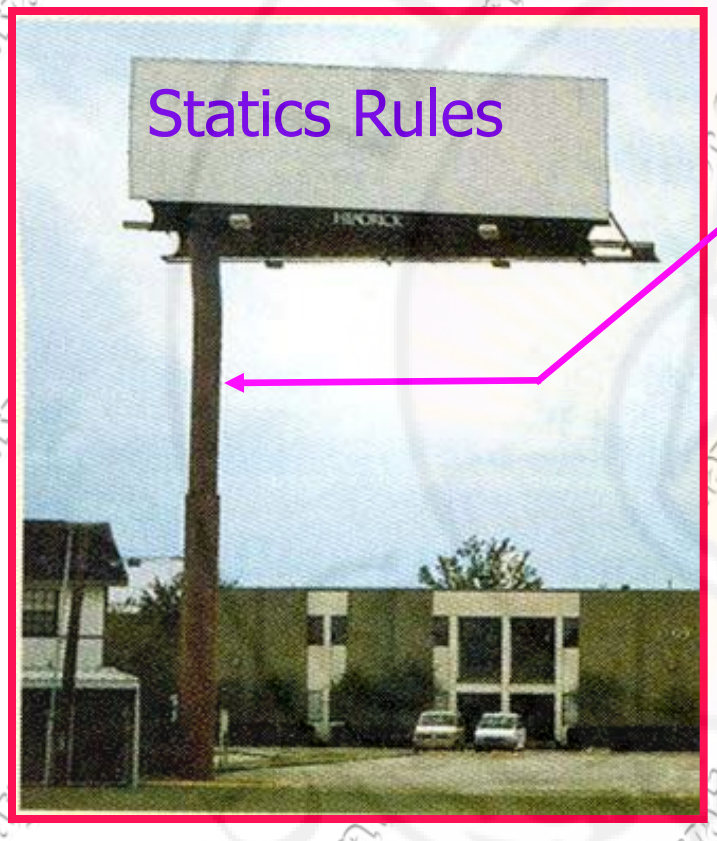
These beams are used to support the roof of this gas station.

Why are the beams tapered? Is it because of the internal forces?

If so, what are these forces and how do we determine them?

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APPLICATIONS



A fixed column supports this rectangular billboard.

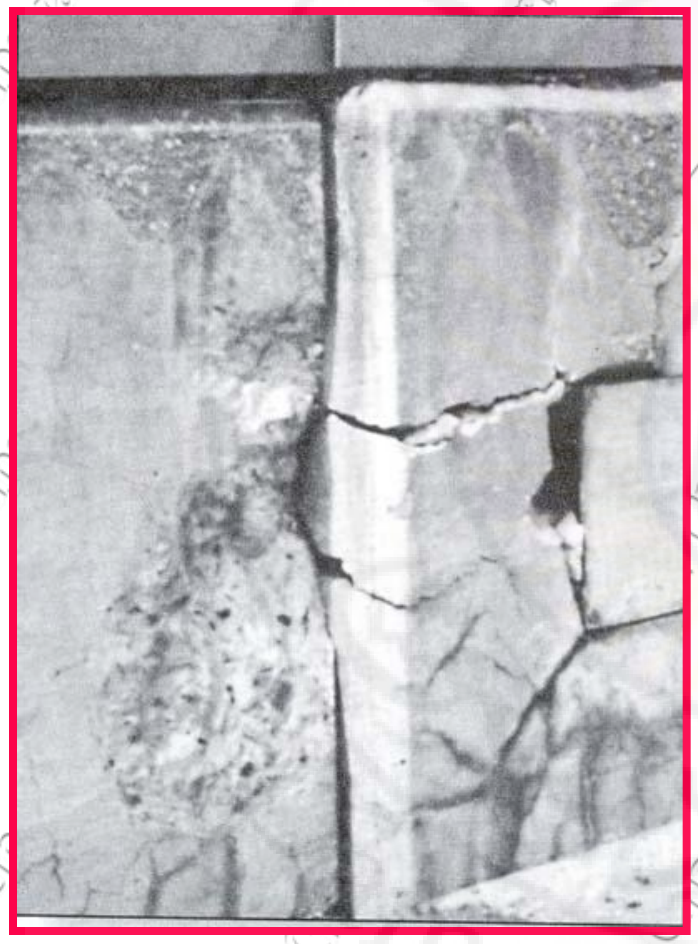
Usually such columns are wider at the bottom than at the top. Why?

Is it because of the internal forces?

If so, what are they and how do we determine them?

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APPLICATIONS



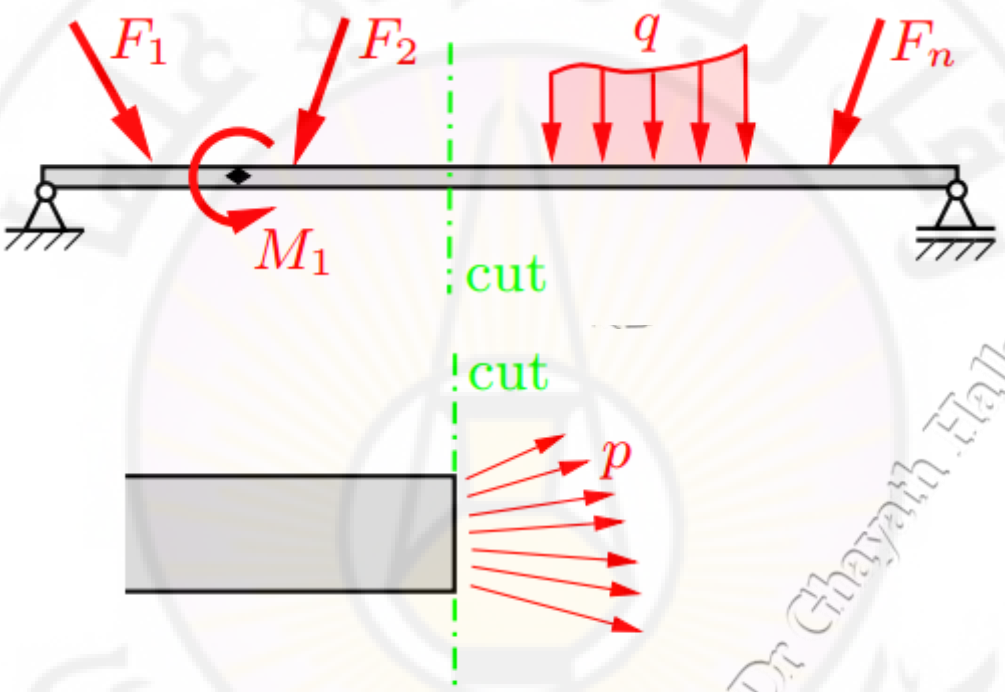
The concrete supporting a bridge has fractured.

What might have caused the concrete to do this?

How can we analyze or design these structures to make them safer?

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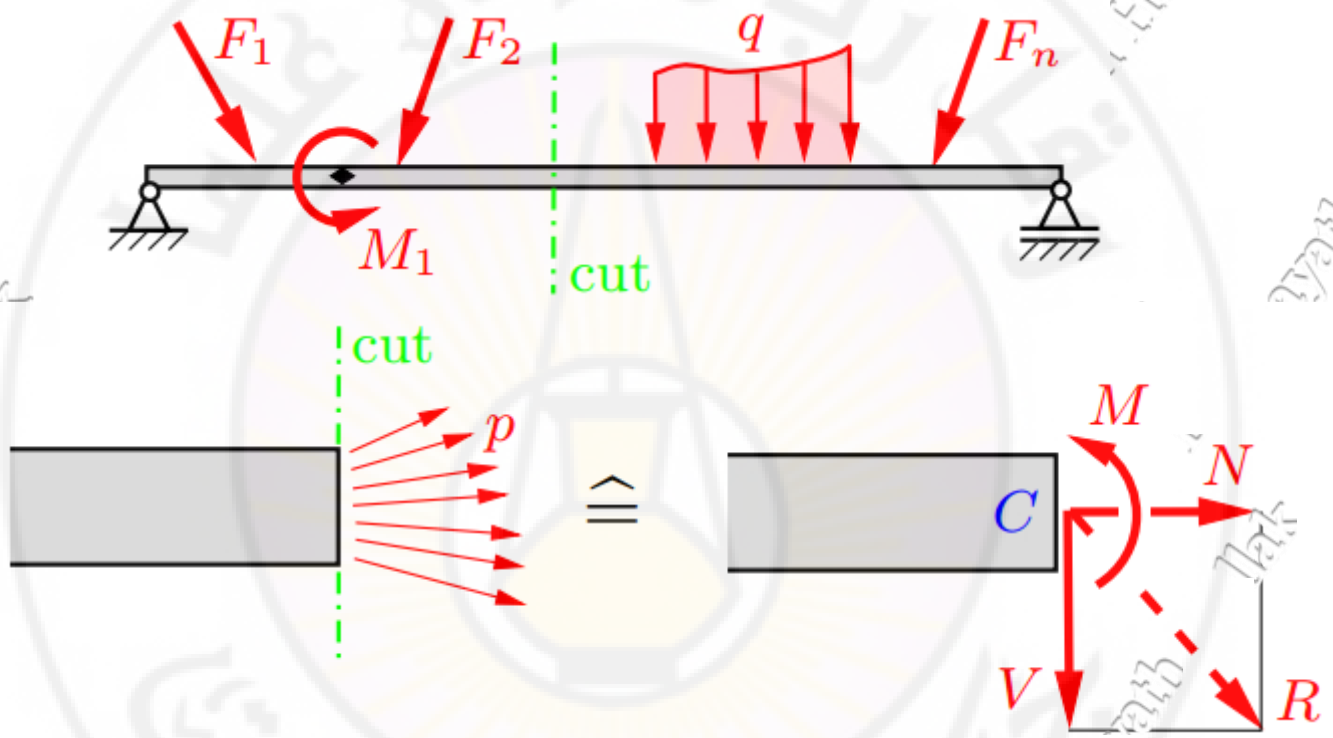
INTERNAL FORCES IN BEAMS



The internal forces in a beam can be made visible and thus accessible to calculation with the aid of a free-body diagram. Accordingly, we pass an imaginary section perpendicularly to the axis of the beam.

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INTERNAL FORCES IN BEAMS

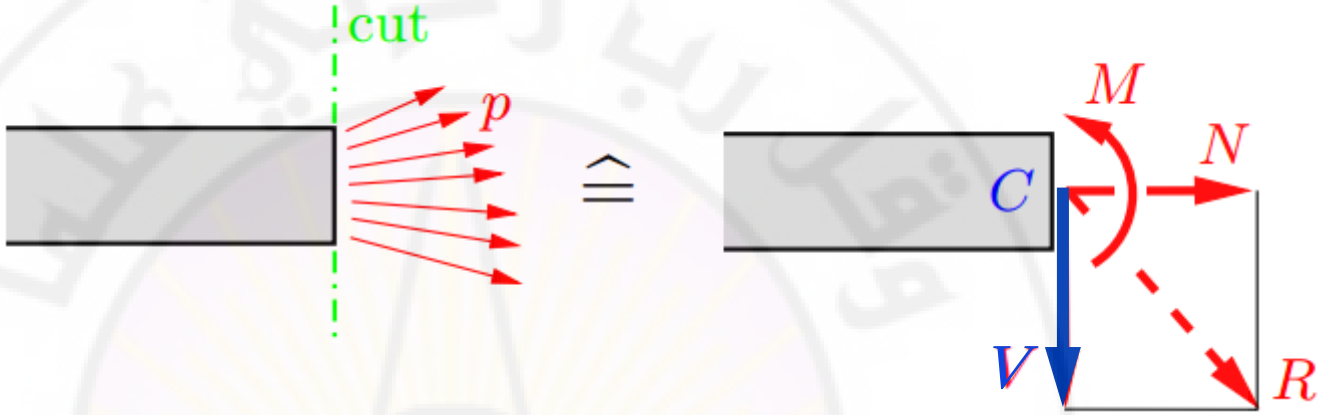


The internal forces p (forces per unit area) acting at the cross-section are distributed across the cross-sectional area. Their intensity is called stress.

it was shown previously that any force system can be replaced by a resultant force R acting at an arbitrary point C

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INTERNAL FORCES IN BEAMS



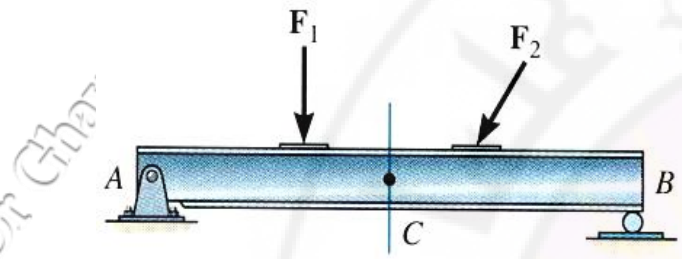
and a corresponding couple $M_{(C)}$. When carrying this out, we choose the centroid C of the cross-sectional area as the reference point of the reduction.

The resultant force *R* is resolved into its components N (normal to the cross-section, in the direction of the axis of the beam) and V (in the cross section, orthogonal to the axis of the beam). The quantities N, V and M are called the stress resultants. In particular,

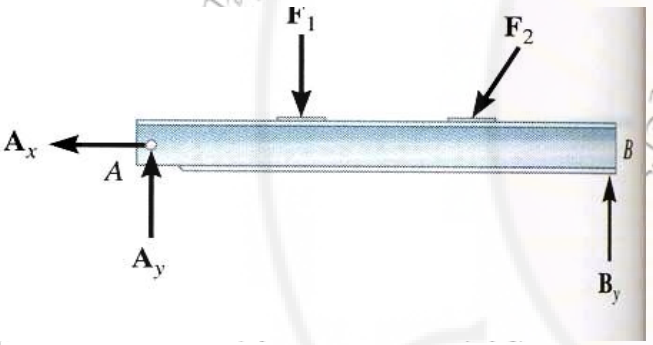
N is called the *normal force*, *V* is the *shear force* and *M* is the *bending moment*

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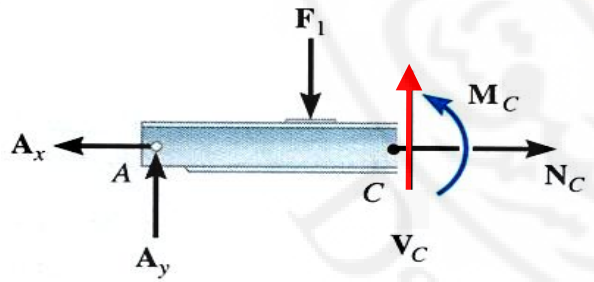
INTERNAL FORCES IN BEAMS



The design of any structural member requires finding the forces acting within the member to make sure the material can resist those loads.

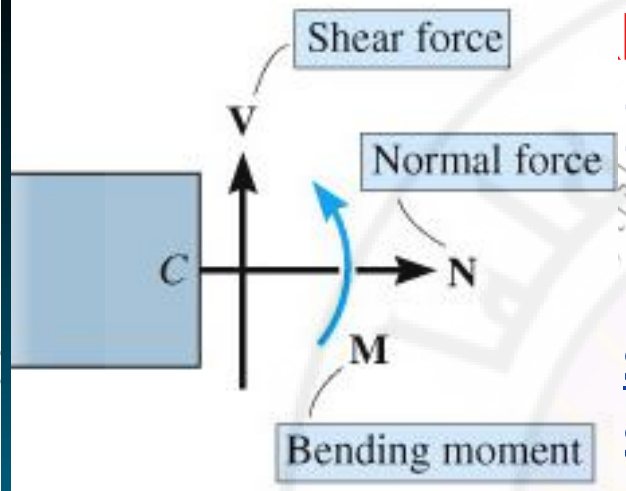


For example, we want to determine the internal forces acting on the cross section at C. First, we first need to determine the support reactions.



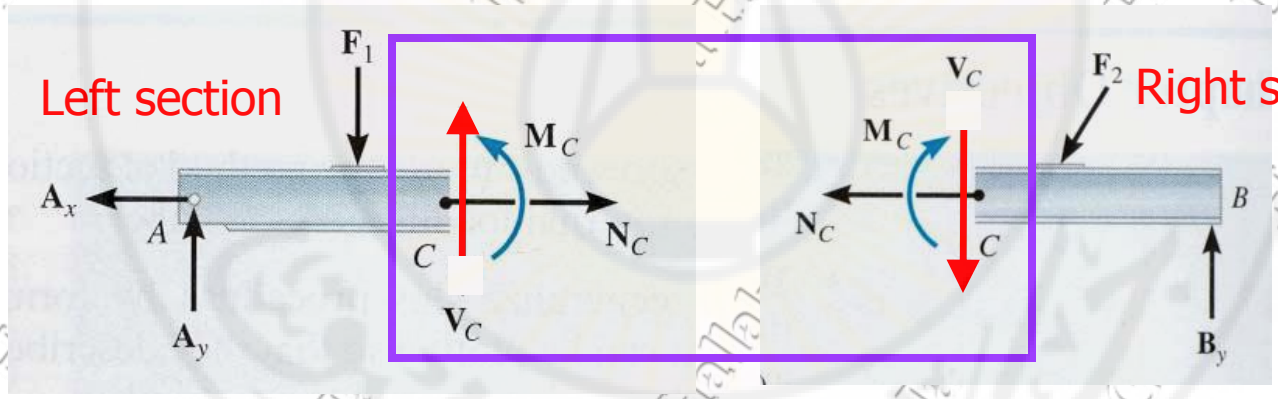
Then we need to cut the beam at C and draw a FBD of one of the halves of the beam. This FBD will include the internal forces acting at C. Finally, we need to solve for these unknowns using the E-of-E.

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NAL FORCES IN BEAMS

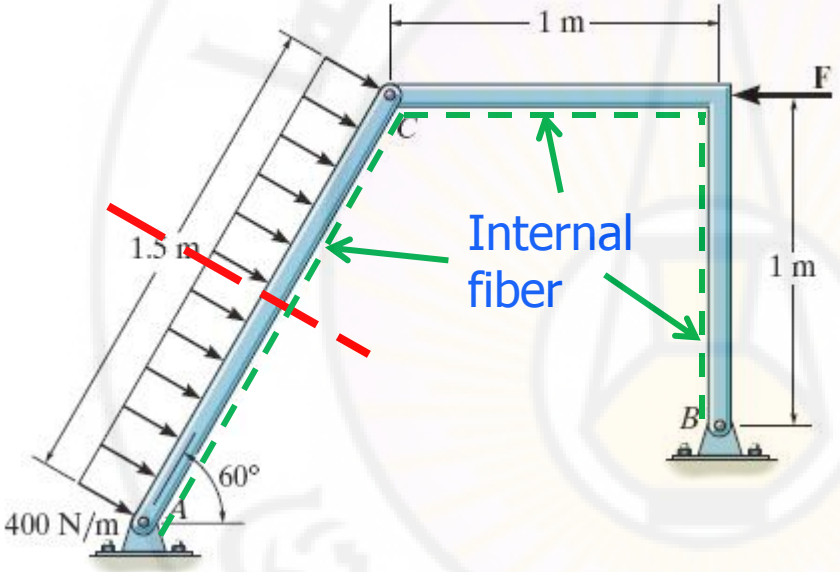
In two-dimensional cases, typical internal loads are normal or axial forces (N, acting perpendicular to the section), shear forces (V, acting along the surface), and the bending moment (M).

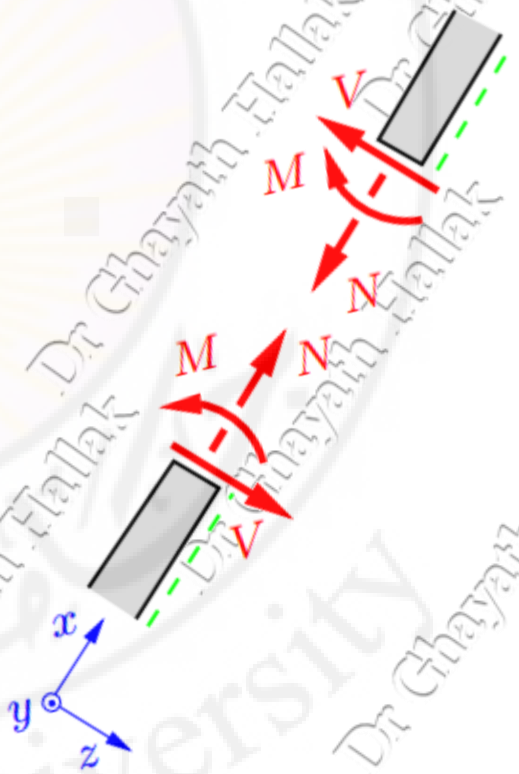


The loads on the left and right sides of the section at C are equal in magnitude but opposite in direction. This is because when the two sides are reconnected, the net loads are zero at the section.

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INTERNAL FORCES IN FRAMES





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STEPS FOR DETERMINING INTERNAL FORCES

1. Take an imaginary cut at the place where you need to determine the internal forces. Then, decide which resulting section or piece will be easier to analyze.

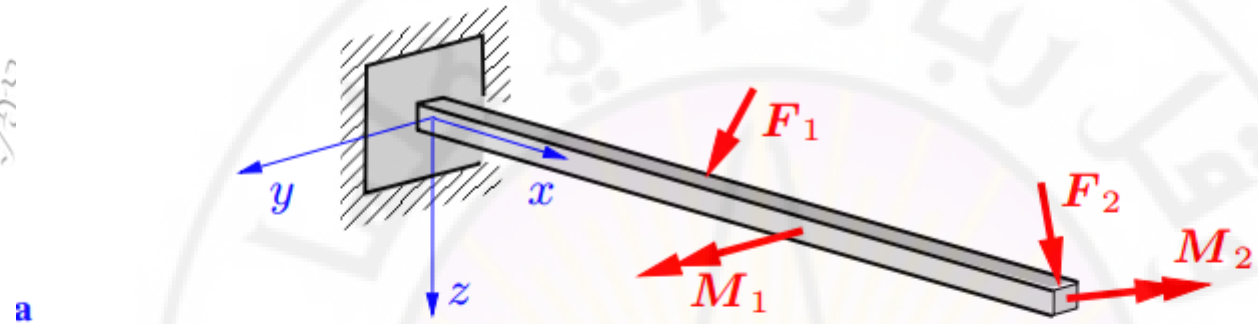
2. If necessary, determine any support reactions or joint forces you need by drawing a FBD of the entire structure and solving for the unknown reactions.

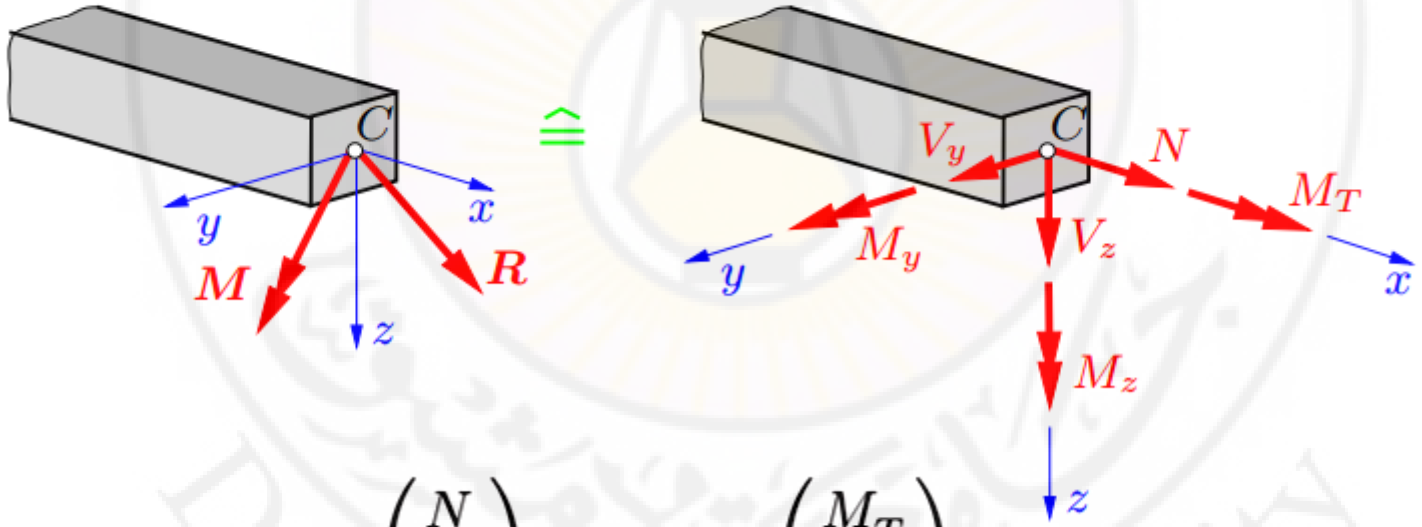
3. Draw a FBD of the piece of the structure you've decided to analyze. Remember to show the N, V, and M loads at the "cut" surface.

4. Apply the E-of-E to the FBD (drawn in step 3) and solve for the unknown internal loads.

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INTERNAL FORCES IN SPATIAL STRUCTURES

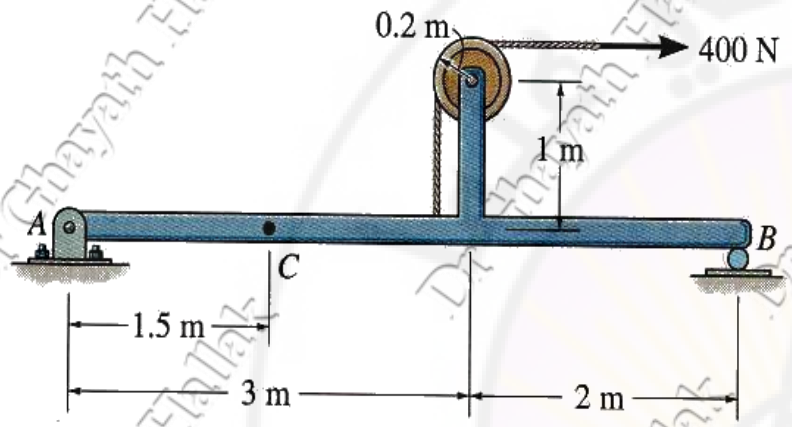




$$\boldsymbol{R} = \begin{pmatrix} N \\ V_y \\ V_z \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} M_T \\ M_y \\ M_z \end{pmatrix}$$

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EXAMPLE



Given: The loading on the beam.

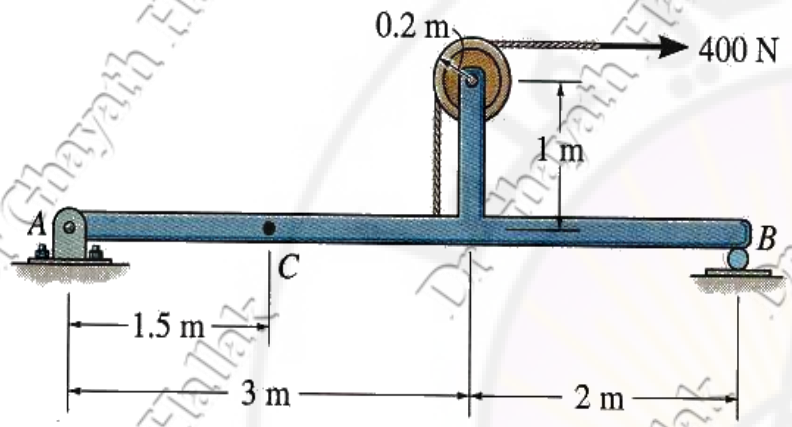
Find: The internal forces at point C.

Solution

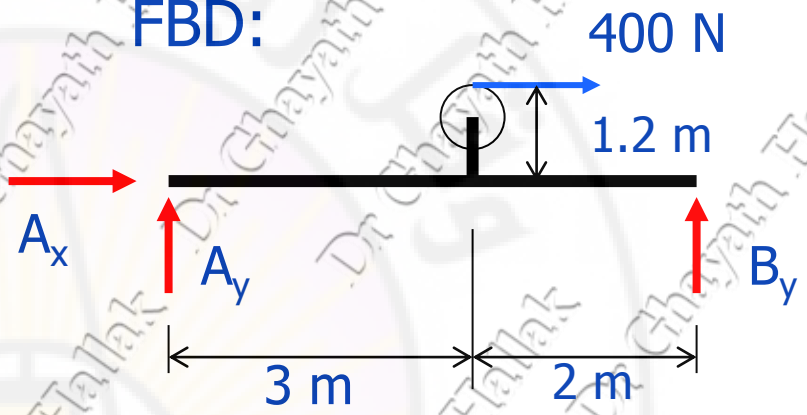
1. Plan on taking the imaginary cut at C. It will be easier to work with the left section (point A to the cut at C) since the geometry is simpler.
2. We need to determine A_x and A_y using a FBD of the entire frame.

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EXAMPLE



FBD:



Applying the E-of-E to this FBD, we get

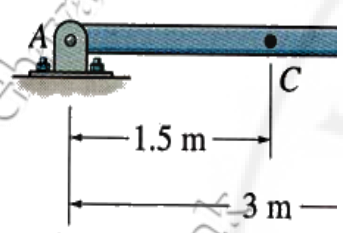
$\rightarrow + \Sigma F_x = A_x + 400 = 0 ; \quad A_x = -400 \text{ N}$

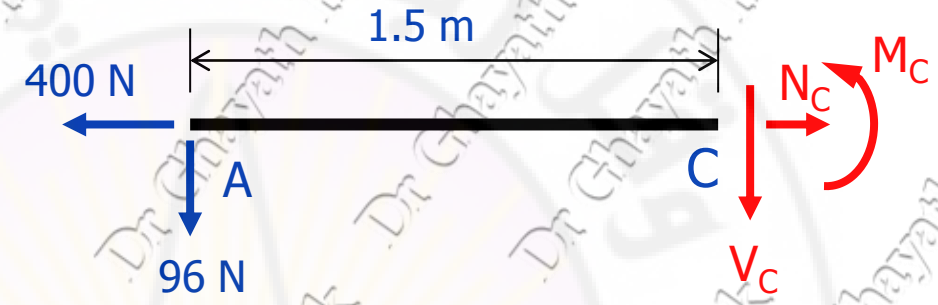
$\circlearrowleft + \Sigma M_B = -A_y(5) - 400\,(1.2) = 0 ; \quad A_y = -96 \text{ N}$

3. Now draw a FBD of the left section. Assume directions for V_C, N_C and M_C.

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EXAMPLE





4. Applying the E of E to this FBD, we get

$\rightarrow + \Sigma F_x = N_C - 400 = 0;$ $\quad N_C = 400\ N$

$\uparrow + \Sigma F_y = -V_C - 96 = 0;$ $\quad V_C = -96N$

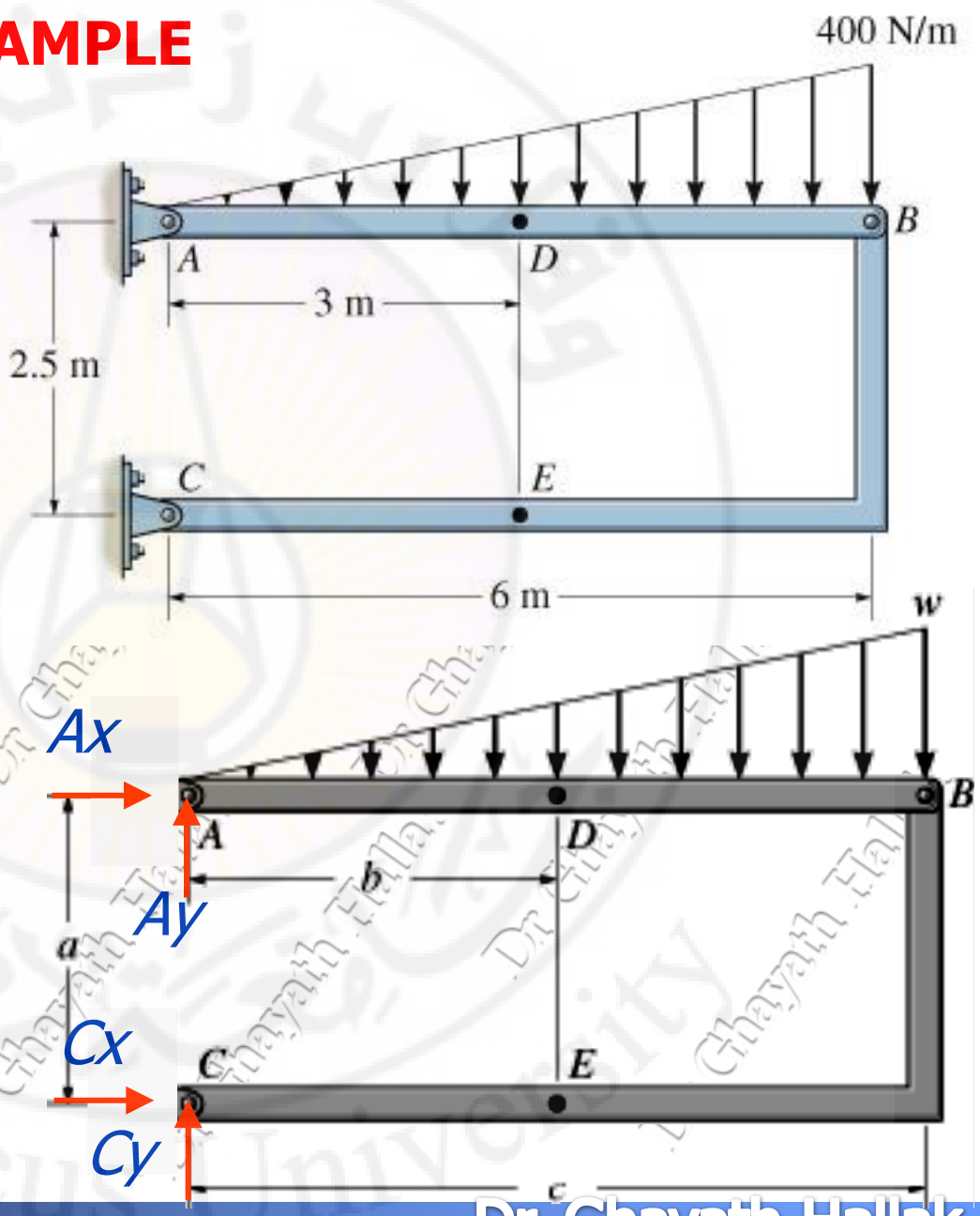
$\circlearrowleft + \Sigma M_C = 96\,(1.5) + M_C = 0\ ;$ $\quad M_C = -144\ N\ m$

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EXAMPLE

Determine the normal force, shear force, and moment at a section passing through point *D* of the two-member frame.

w =400N/m, *a* = 2.5 m, *b* = 3 m, *c* = 6 m



Solution:

$\curvearrowleft \Sigma M_C = 0;$

$Ax \times 2.5 + (400 \times 6/2) \times (2 \times 6/3) = 0$

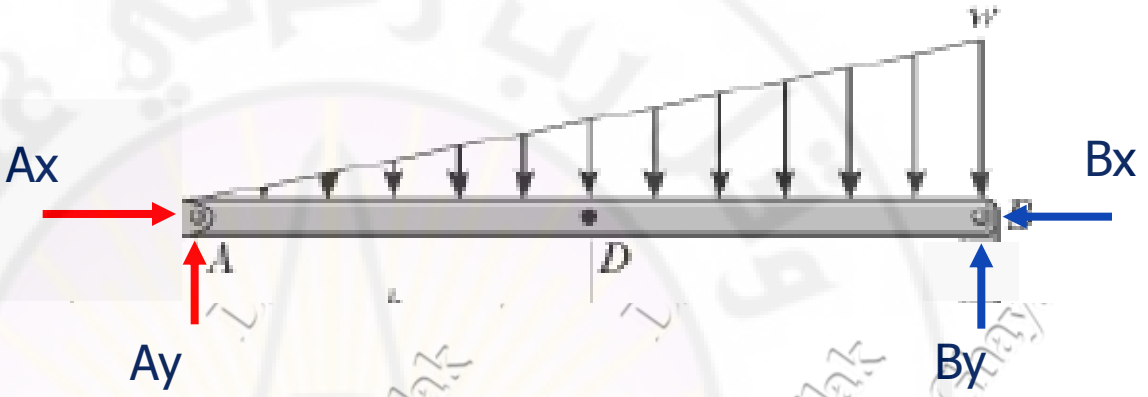
$Ax = -1920$ N

$\rightarrow + \Sigma Fx = 0;\ Cx + Ax = 0$

$Cx = -Ax = 1920$ N

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EXAMPLE



$\circlearrowleft \Sigma M_A = 0;$ ⟹ By ×6 – (400×6/2) ×(2×6/3)=0

By = 800 N

$\uparrow + \Sigma Fy = 0;$ ⟹ Ay + 800 – (400×6/2) =0

Ay = 400 N

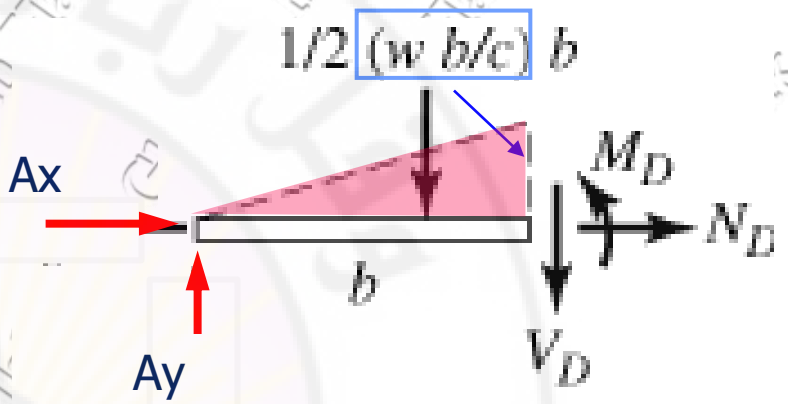
$\rightarrow + \Sigma Fx = 0;$ ⟹ Ax – Bx =0

Bx = Ax = – 1920 N

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EXAMPLE

Internal Forces at point D:



$\rightarrow + \Sigma Fx = 0;\ N_D + Ax = 0$

$N_D = -Ax \Rightarrow N_D = 1920$ N

$\uparrow + \Sigma Fy = 0;$

$Ay - V_D - 1/2 \times (400 \times 3/6) \times 3 = 0$

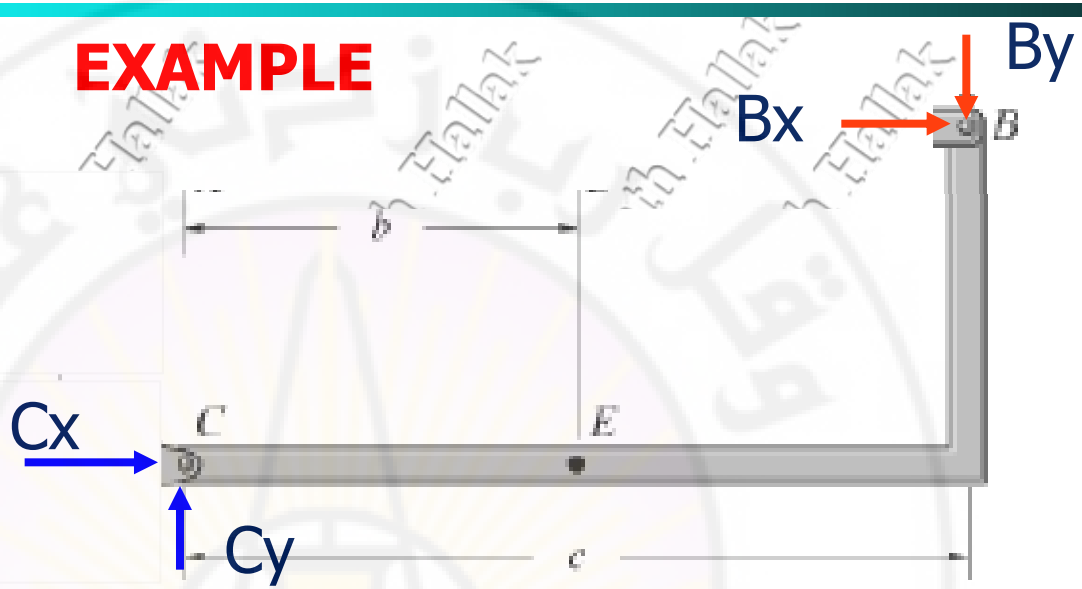
$\Rightarrow V_D = 100$ N

$\circlearrowleft \Sigma M_D = 0;\quad -Ay \times 3 + (400 \times 3/6) \times (3/2 \times 1/3 \times 3) + M_D = 0$

$\Rightarrow M_D = 900$ N.m

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EXAMPLE



$\rightarrow + \Sigma Fx = 0;\ \ Cx + Bx = 0 \Rightarrow Cx = -Bx = 1920$ N

$\uparrow + \Sigma Fy = 0;\ \ Cy - By = 0 \Rightarrow Cy = By = 800$ N

$\rightarrow + \Sigma Fx = 0;\ \ Cx + N_E = 0 \Rightarrow$

$N_E = -Cx = -1920$ N

$\uparrow + \Sigma Fy = 0;\ \ Cy - VE = 0 \Rightarrow$

$VE = Cy = 800$ N



$\circlearrowleft \Sigma M_E = 0;\ \ -Cy \times 3 + M_E = 0$

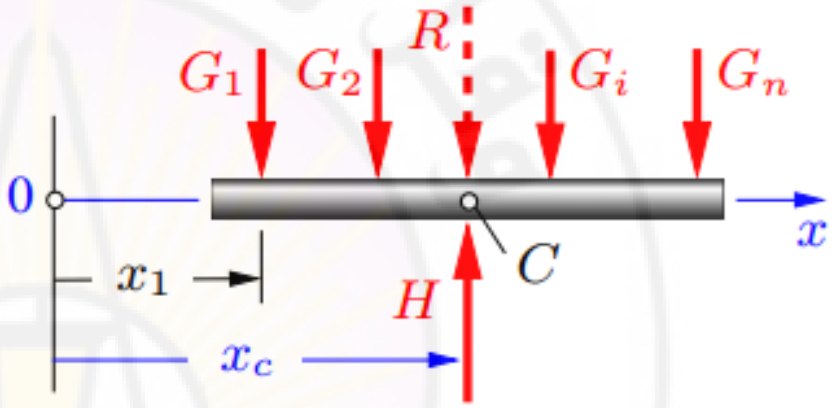
$M_E = 2400$ N.m

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

1 Center of Forces

Applying the equilibrium conditions



$$\uparrow: \quad H - \sum G_i = 0, \qquad \circlearrowleft 0: \quad x_c H - \sum x_i G_i = 0$$

$$x_c = \frac{\sum x_i G_i}{\sum G_i}$$

The corresponding point C (an arbitrary point on the action line of H) is called the *center of forces*.

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

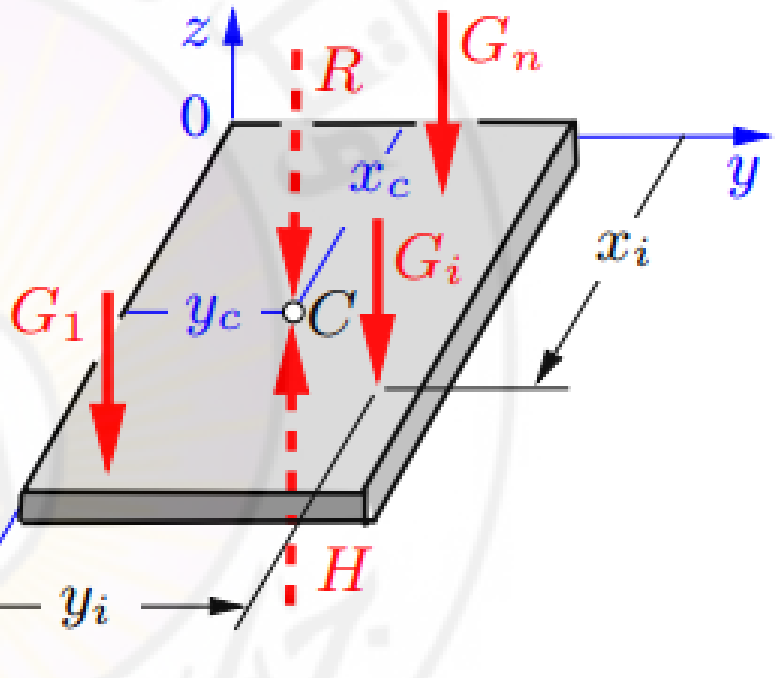
1 Center of Forces

Applying the equilibrium conditions

$$\sum F_{iz} = 0: \qquad H - \sum G_i = 0,$$

$$\sum M_{ix}^{(0)} = 0: \qquad y_c H - \sum y_i G_i = 0,$$

$$\sum M_{iy}^{(0)} = 0: \qquad -x_c H + \sum x_i G_i = 0$$



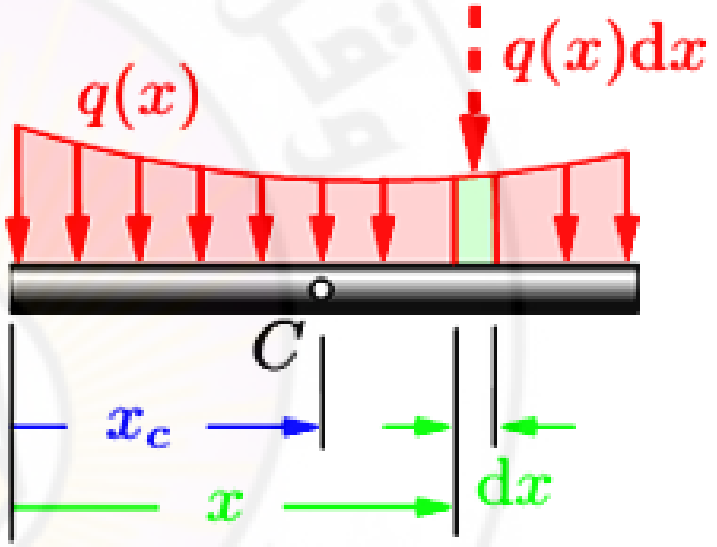
$$x_c = \frac{\sum x_i G_i}{\sum G_i}, \qquad y_c = \frac{\sum y_i G_i}{\sum G_i}.$$

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

1 Center of Forces

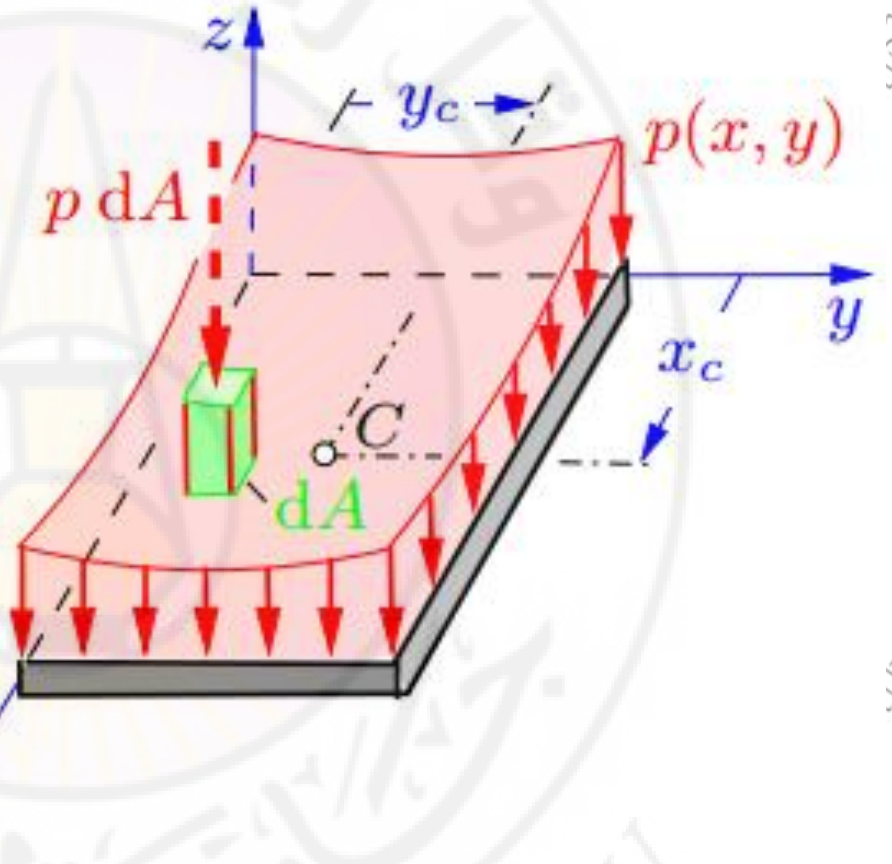
$$x_c = \frac{\int x\, q(x)\, \mathrm{d}x}{\int q(x)\, \mathrm{d}x}.$$



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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

1 Center of Forces



$$x_c = \frac{\int x\, p(x,y)\, \mathrm{d}A}{\int p(x,y)\, \mathrm{d}A}, \qquad y_c = \frac{\int y\, p(x,y)\, \mathrm{d}A}{\int p(x,y)\, \mathrm{d}A}.$$

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

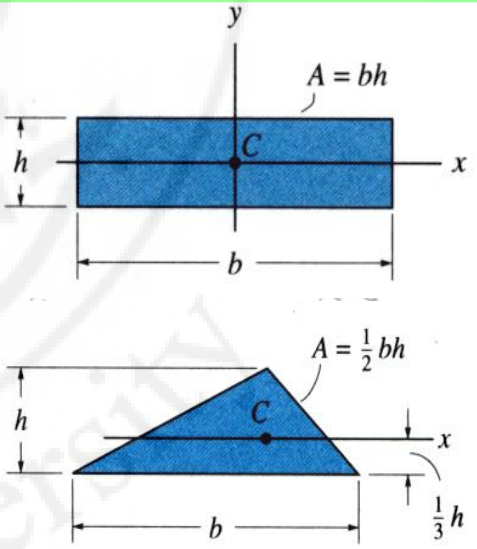
2 Center of Gravity and Center of Mass

The center of gravity (G) is a point which locates the resultant weight of a system of particles or body.

The center of mass is a point which locates the resultant mass of a system of particles or body. Generally, its location is the same as that of G.

The centroid C is a point which defines the geometric center of an object.

The centroid coincides with the center of mass or the center of gravity only if the material of the body is homogenous (density or specific weight is constant throughout the body).



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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

2 Center of Gravity and Center of Mass

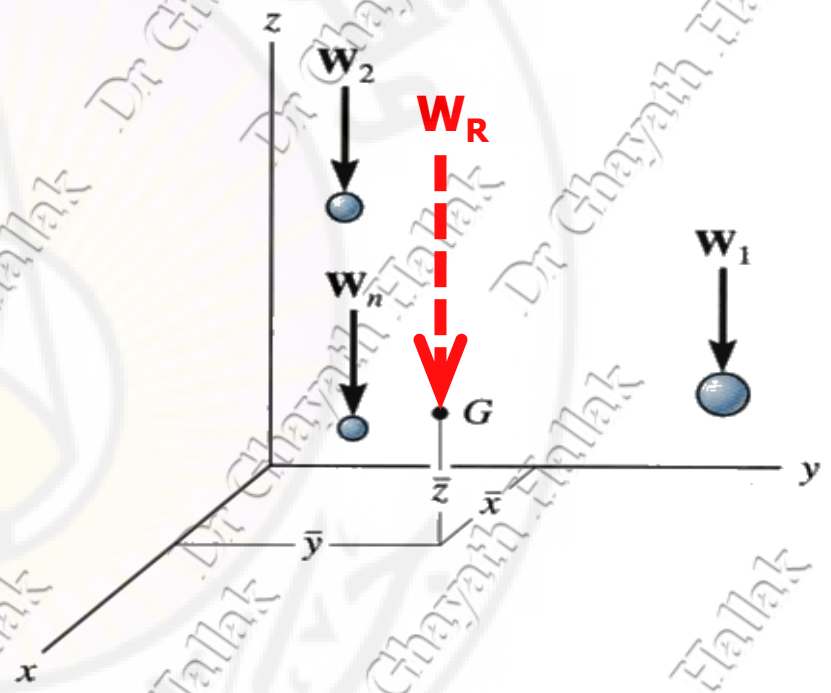
Consider a system of n particles as shown in the figure. The net or the resultant weight is given as $W_R = \Sigma W$.

Summing the moments about the y-axis, we get

$$\bar{x} W_R = x_1 W_1 + x_2 W_2 + \cdots x_n W_n$$

where x_1 represents x coordinate of W_1, etc..

Similarly, we can sum moments about the x- and z-axes to find the coordinates of G.



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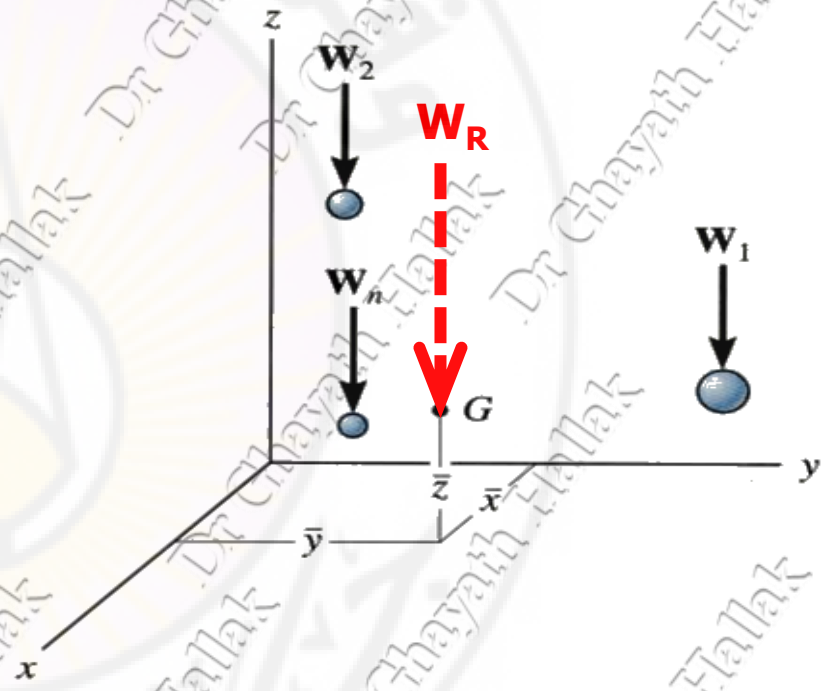
CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

2 Center of Gravity and Center of Mass

coordinates of the center of gravity

$$\bar{x} = \frac{\sum \tilde{x} W}{\sum W}; \bar{y} = \frac{\sum \tilde{y} W}{\sum W},$$

$$\bar{z} = \frac{\sum \tilde{z} W}{\sum W}$$



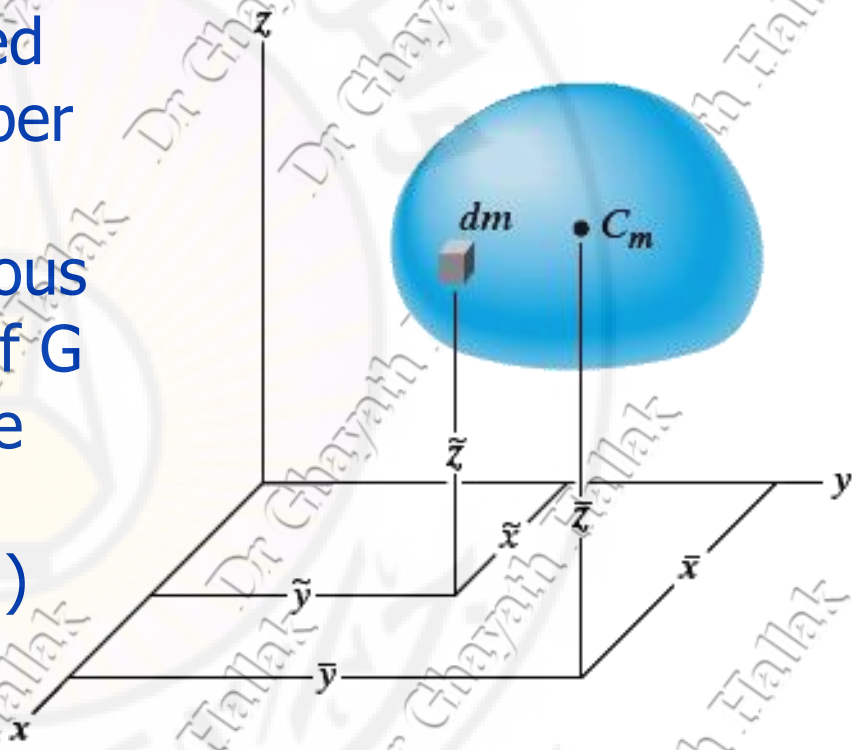
By replacing the W with a M in these equations, the coordinates of the center of mass can be found.

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

2 Center of Gravity and Center of Mass

A **rigid body** can be considered as made up of an infinite number of particles. Hence, using the same principles as in the previous slide, we get the coordinates of G by simply replacing the discrete summation sign (Σ) by the continuous summation sign (∫) and W by dW.



$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW}; \bar{y} = \frac{\int \tilde{y}\,dW}{\int dW}; \bar{z} = \frac{\int \tilde{z}\,dW}{\int dW}$$

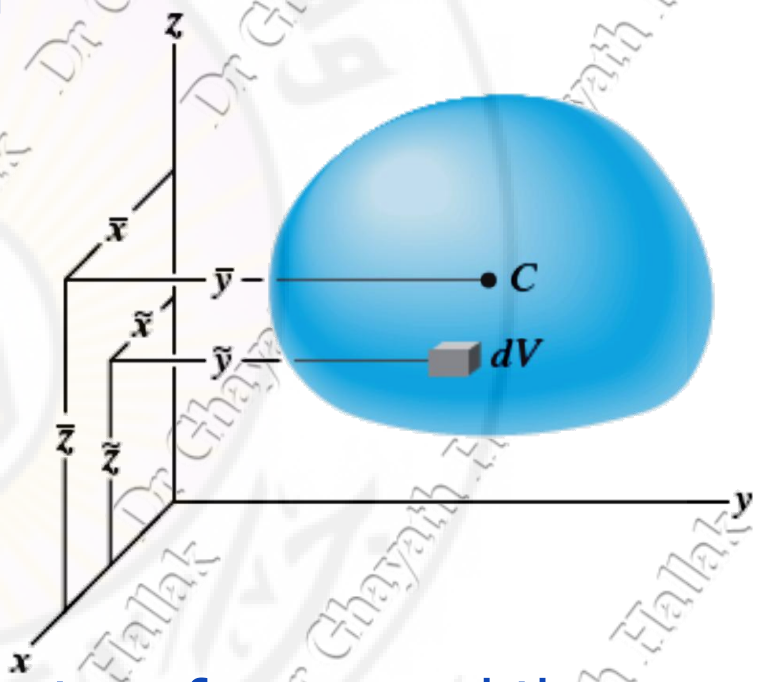
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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

3 Center of Volume

- Consider an object subdivided into volume elements dV, for location of the centroid,

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV};\quad \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV};\quad \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV}$$

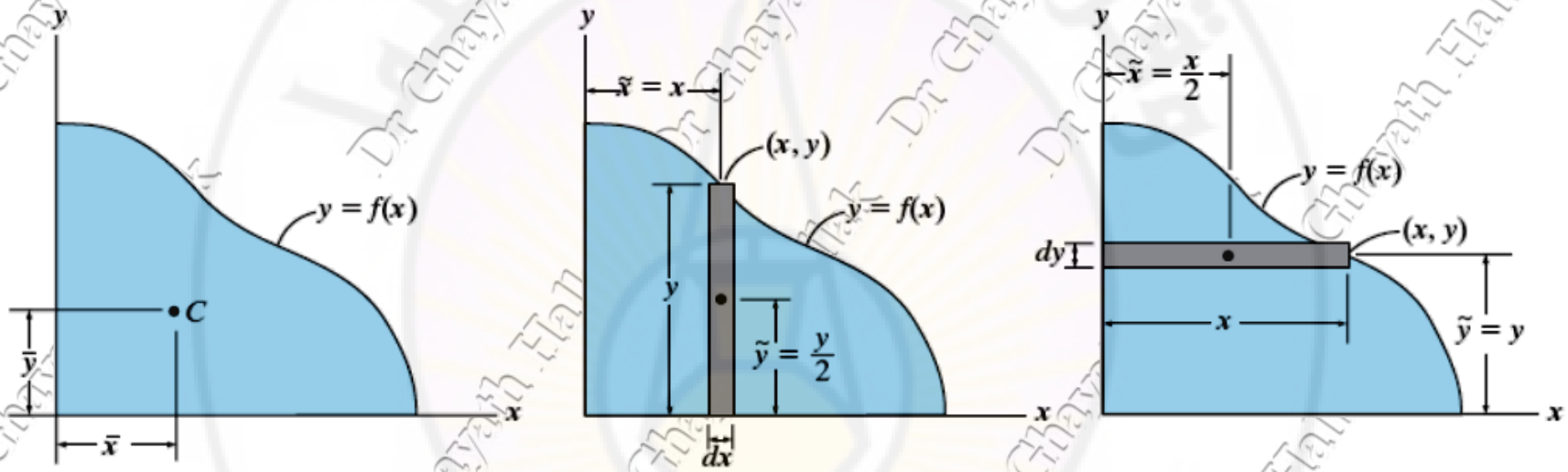


Similarly, the coordinates of the center of mass and the centroid of area, or length can be obtained by replacing V by m, A, or L, respectively.

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of an Area



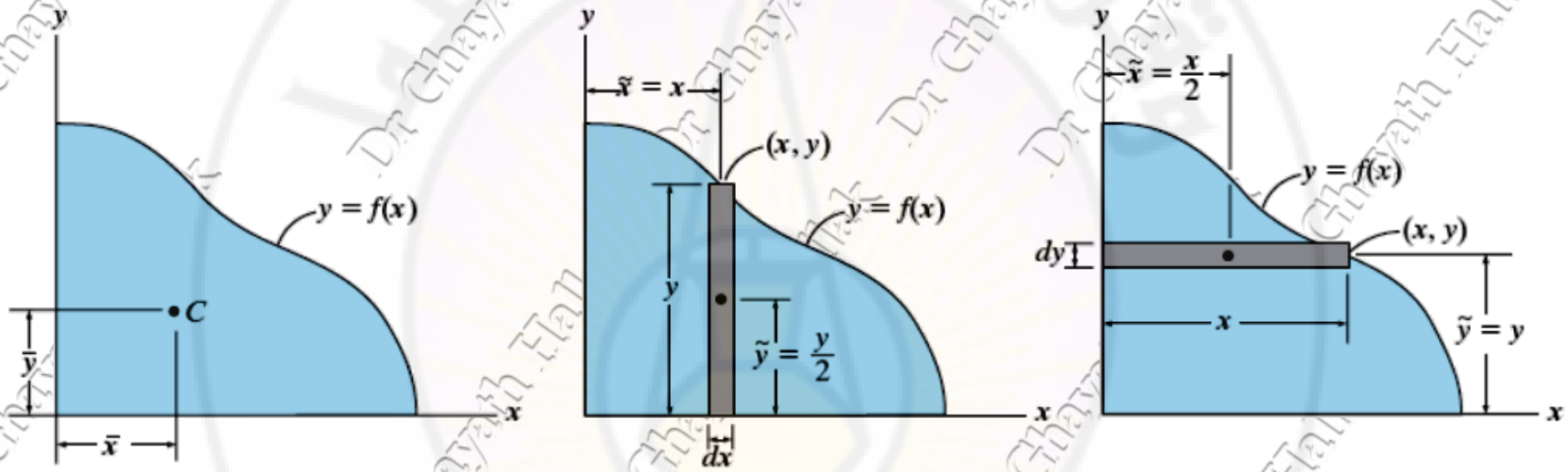
For centroid for surface area of an object, such as plate and shell, subdivide the area into differential elements dA

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA}\,;\ \bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA}$$

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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of an Area



$$Q_x = \int_A \tilde{y}\ dA$$

$$Q_y = \int_A \tilde{x}\ dA$$

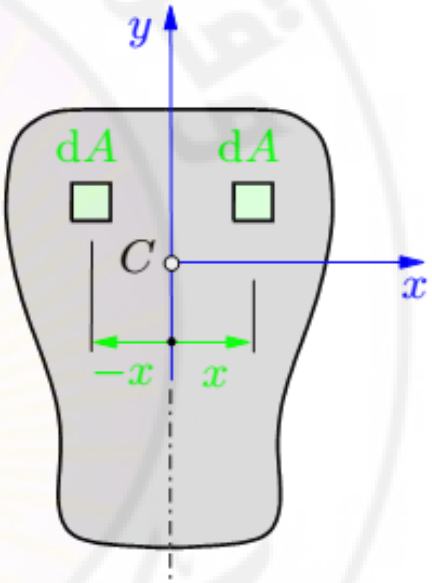
The integrals are called the first moments of the area with respect to the x and the y-axis, respectively:

The first moments of an area with respect to axes through its centroid are zero.

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4 Centroid of an Area



If the area has an axis of symmetry, the centroid of the area lies on this axis.

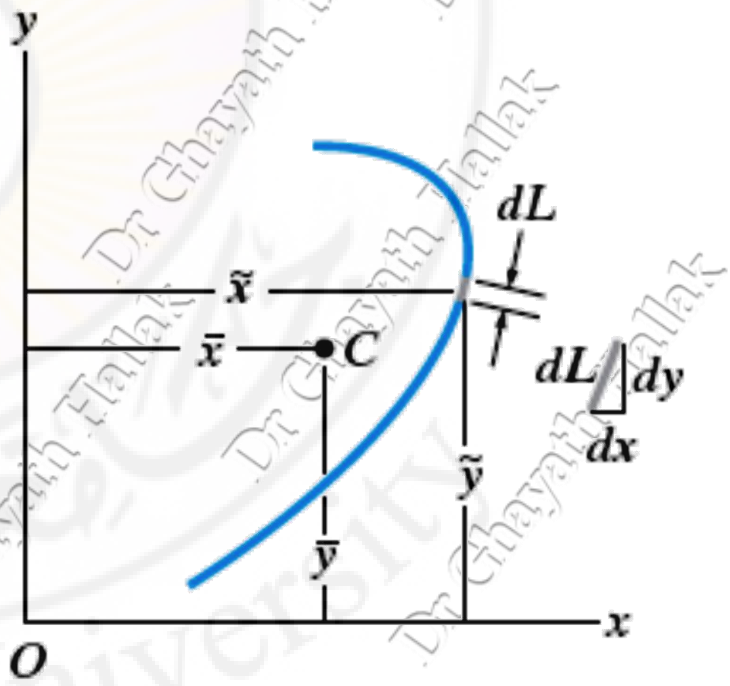
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CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of a Line

- If the geometry of the object takes the form of a line, the balance of moments of differential elements dL about each of the coordinate system yields

$$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL}; \quad \bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL}; \quad \bar{z} = \frac{\int_L \tilde{z}\,dL}{\int_L dL}$$



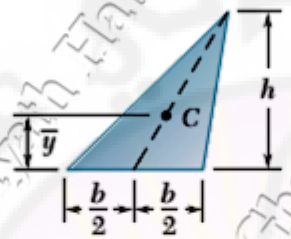
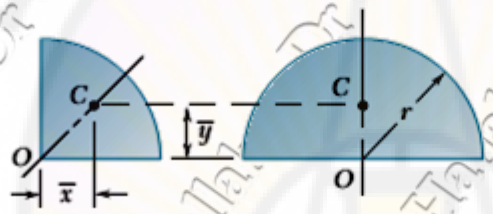
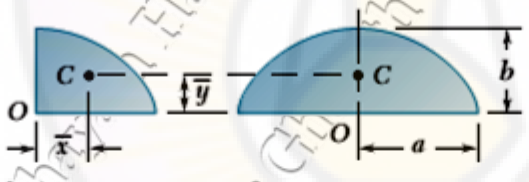
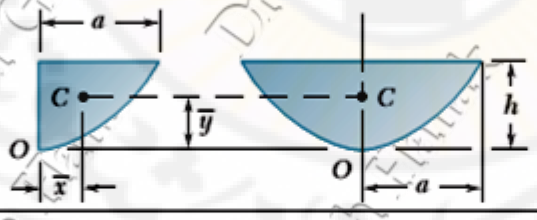
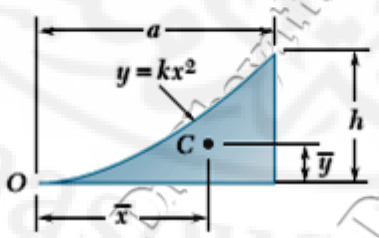
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STEPS FOR DETERMINING AREA CENTROID

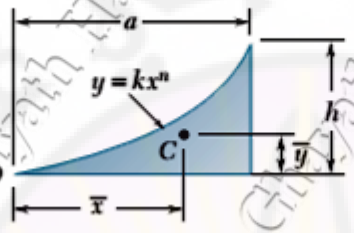
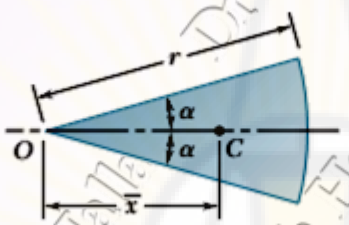
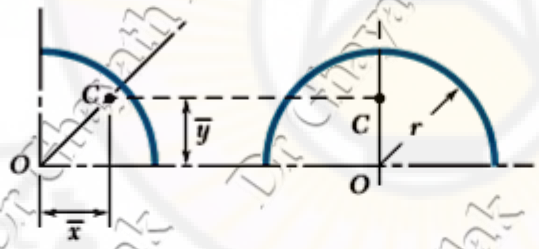
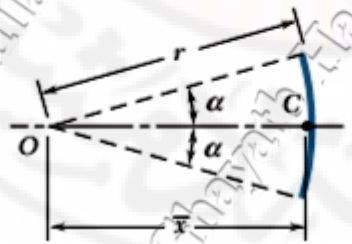
1. Choose an appropriate differential element dA at a general point (x,y). Hint: Generally, if y is easily expressed in terms of x (e.g., $y = x^2 + 1$), use a vertical rectangular element. If the converse is true, then use a horizontal rectangular element.
2. Express dA in terms of the differentiating element dx (or dy).
3. Determine coordinates $(\tilde{x}, \tilde{y})$ of the centroid of the rectangular element in terms of the general point (x,y).
4. Express all the variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

Note: Similar steps are used for determining CG, CM, etc..
These steps will become clearer by doing a few examples.

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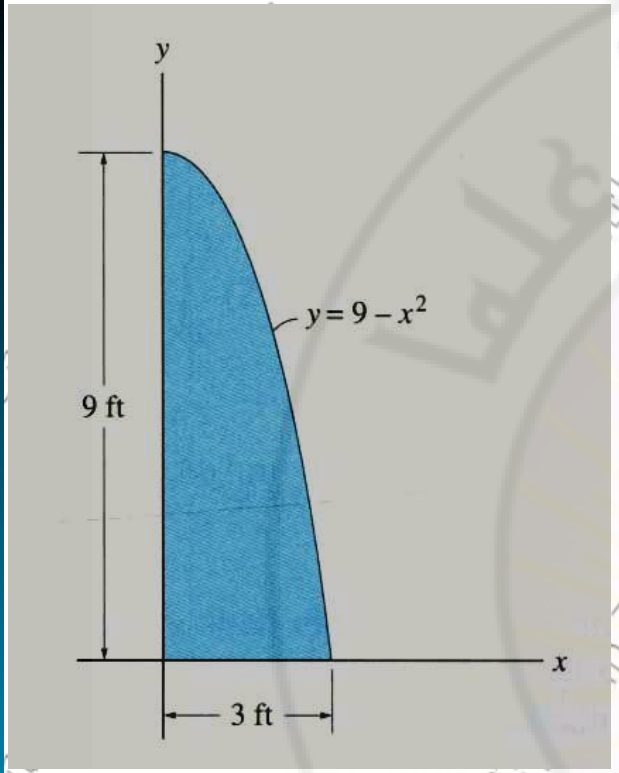
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

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Shape		$\bar{x}$	$\bar{y}$	Area
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

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EXAMPLE



Given: The area as shown.

Find: The centroid location (x , y)

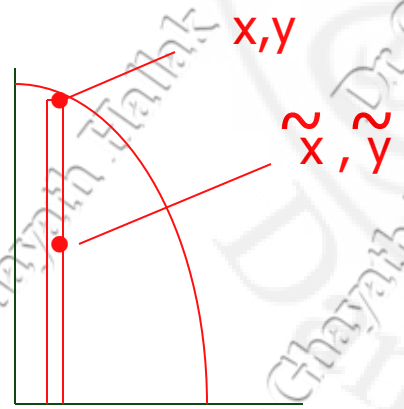
Plan: Follow the steps.

Solution

1. Since y is given in terms of x, choose dA as a vertical rectangular strip.

2. $dA = y\,dx = (9 - x^2)\,dx$

3. $\tilde{x} = x$ and $\tilde{y} = y / 2$

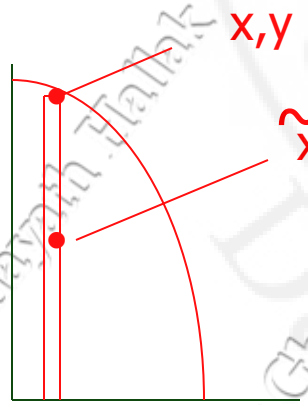


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EXAMPLE

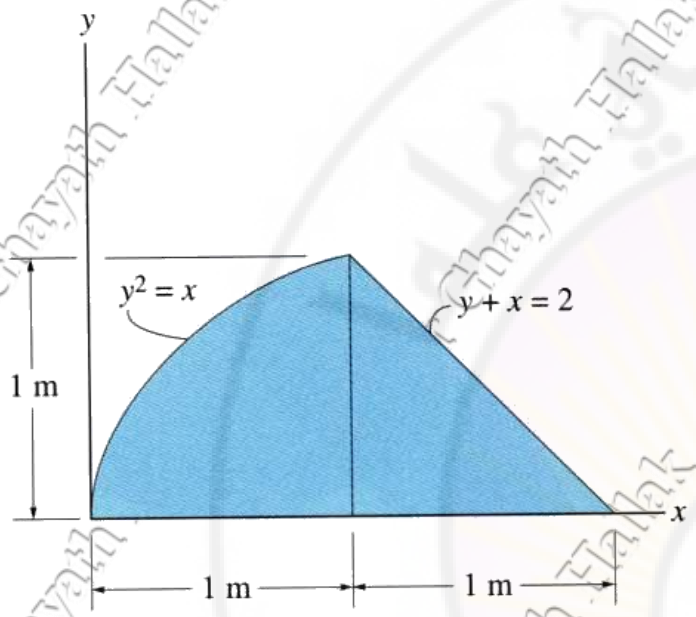
$$\bar{x} = \frac{\int \tilde{x}\,dA}{\int dA} = \frac{\int_0^3 x\left(9 - x^2\right)dx}{\int_0^3 \left(9 - x^2\right)dx} = \frac{\left[9\frac{x^2}{2} - \frac{x^4}{4}\right]_0^3}{\left[9x - \frac{x^3}{3}\right]_0^3} = \frac{20.25}{18} = 1.125$$

$$\bar{y} = \frac{\int \tilde{y}\,dA}{\int dA} = 0.5\frac{\int_0^3 \left(9 - x^2\right)\left(9 - x^2\right)dx}{\int_0^3 \left(9 - x^2\right)dx} = 0.5\frac{\left[81x - \frac{18x^3}{3} + \frac{x^5}{5}\right]_0^3}{\left[9x - \frac{x^3}{3}\right]_0^3} = 0.5\frac{129.6}{18} = 3.6$$



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EXAMPLE

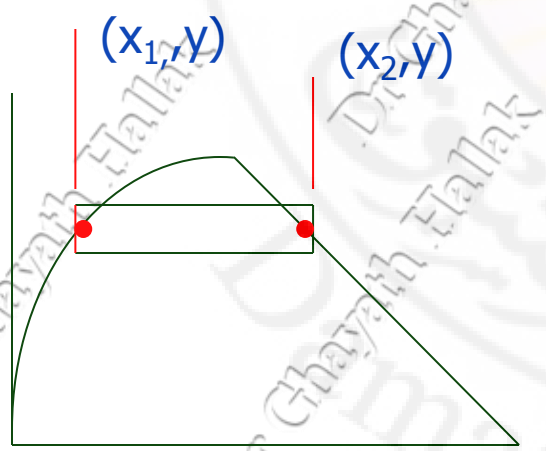


Given: The area as shown.

Find: The x of the centroid.

Plan: Follow the steps.

Solution



1. Choose dA as a horizontal rectangular strip.

2. $dA = (x_2 - x_1)\, dy$

   $= ((2 - y) - y^2)\, dy$

3. $\tilde{x} = (x_1 + x_2) / 2$

   $= 0.5 ((2 - y) + y^2)$

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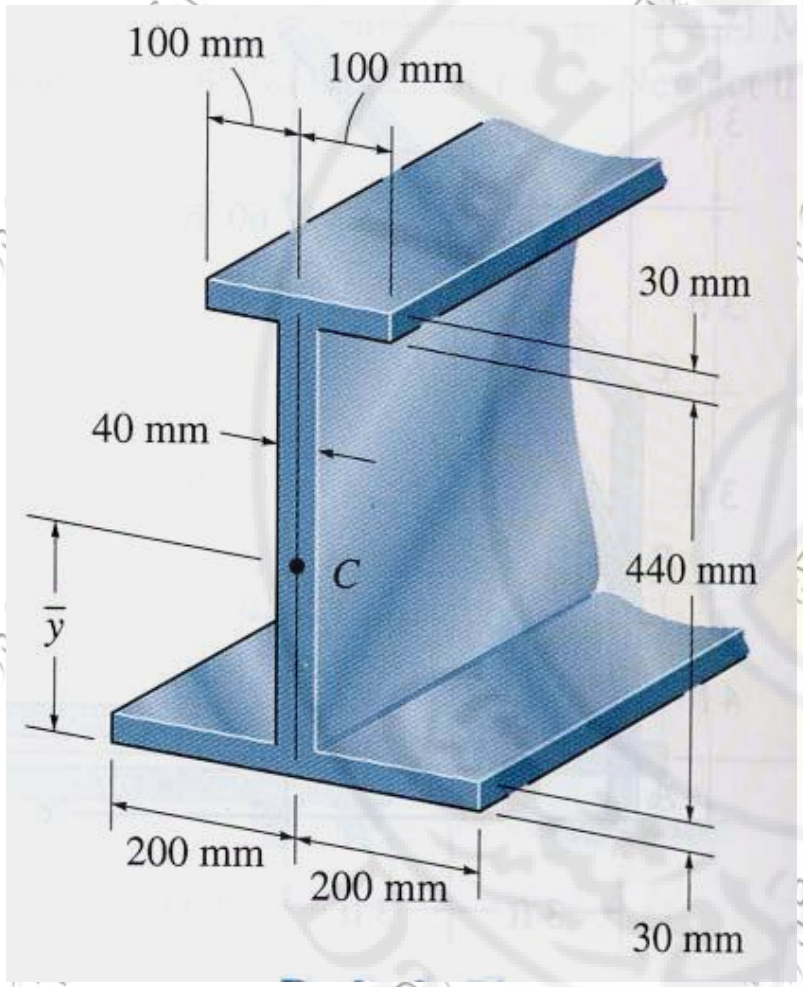
$$\bar{x} = \frac{\int \tilde{x}\,dA}{\int dA} = \frac{\int_0^1 0.5\left(2 - y + y^2\right)\left(2 - y - y^2\right)dy}{\int_0^1 \left(2 - y - y^2\right)dy} = \frac{\int_0^1 0.5\left(4 - 4y + y^2 - y^4\right)dy}{\int_0^1 \left(2 - y - y^2\right)dy}$$

$$= 0.5\frac{\left[4y - 4\frac{y^2}{2} + \frac{y^3}{3} - \frac{y^5}{5}\right]_0^1}{\left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1} = 0.5\frac{2.13}{1.167} = 0.91$$

$$\bar{y} = \frac{\int \tilde{y}\,dA}{\int dA} = \frac{\int_0^1 y\left(2 - y - y^2\right)dy}{\int_0^1 \left(2 - y - y^2\right)dy} = \frac{\left[2\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1}{\left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1} = \frac{0.416}{1.167} = 0.357$$

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COMPOSITE BODIES





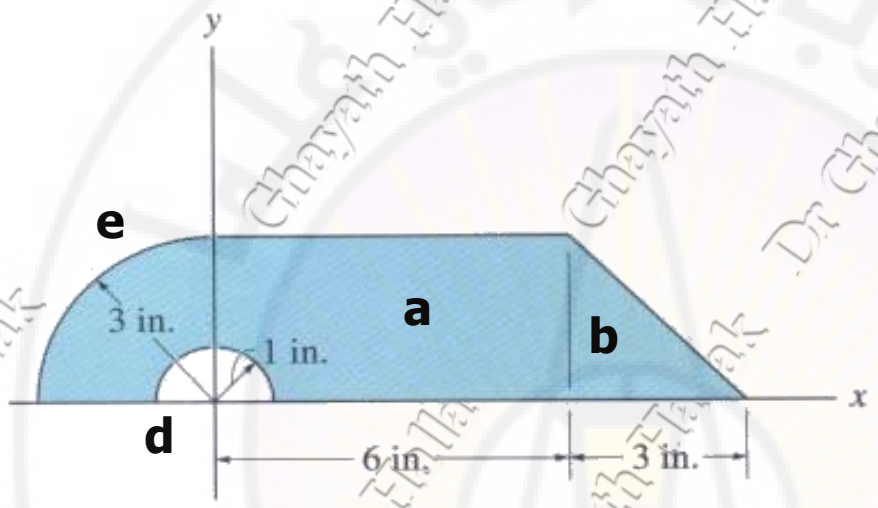
The I-beam is commonly used in building structures.

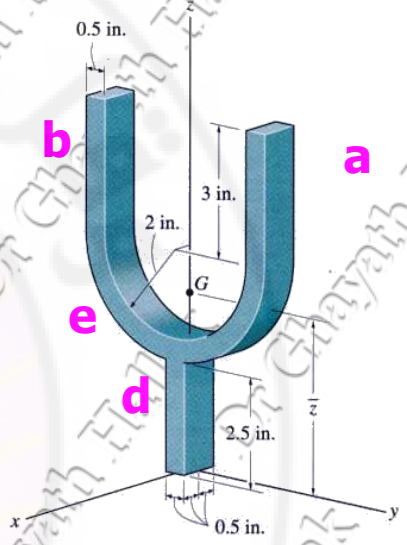
When doing a stress analysis on an I - beam, the location of the centroid is very important.

How can we <u>easily</u> determine the location of the centroid for a given beam shape?

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CONCEPT OF A COMPOSITE BODY



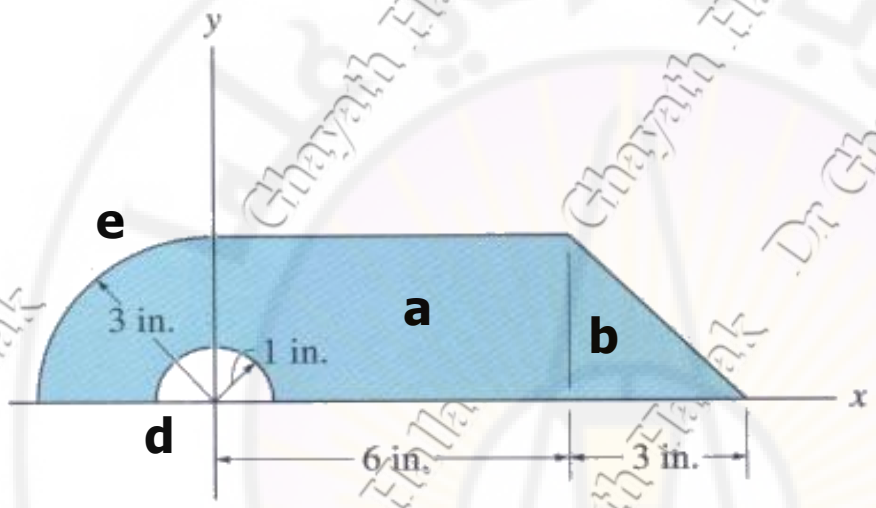


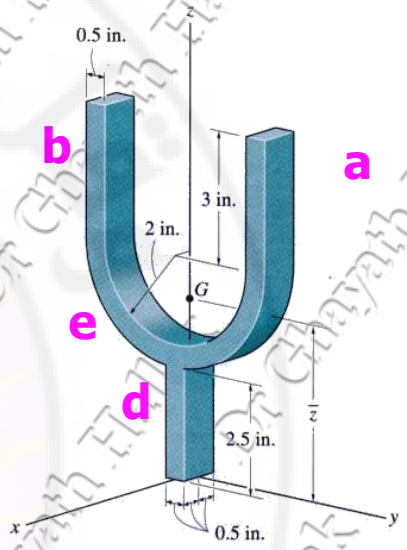
Many industrial objects & construction members can be considered as composite bodies **made up of a series of connected "simpler" shaped parts or holes**, like a rectangle, triangle, and semicircle.

Knowing the location of the centroid, C, or center of gravity, G, of the simpler shaped parts, we can easily determine the location of the C or G for the more complex composite body.

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CONCEPT OF A COMPOSITE BODY





This can be done by considering each part as a "particle" and following the procedure as described earlier.

This is a **simple, effective, and practical method** of determining the location of the centroid or center of gravity for composite body or surface.

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STEPS FOR ANALYSIS

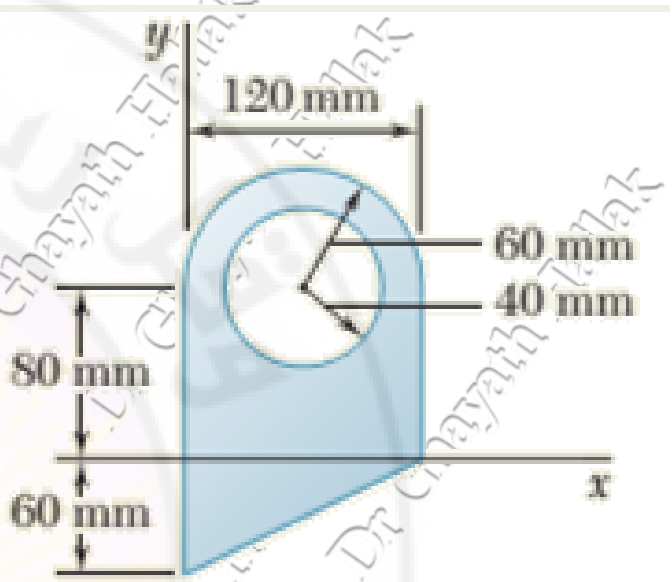
1. Divide the body into pieces that are known shapes. Holes are considered as pieces with negative weight or size.
2. Make a table with the first column for segment number, the second column for weight, mass, or size (depending on the problem), the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.
3. Fix the coordinate axes, determine the coordinates of the center of gravity of centroid of each piece, and then fill-in the table.
4. Sum the columns to get x, y, and z. Use formulas like
5. $$X_c = (\Sigma \tilde{x}_i A_i) / (\Sigma A_i) \quad \text{or} \quad X_c = (\Sigma \tilde{x}_i W_i) / (\Sigma W_i)$$

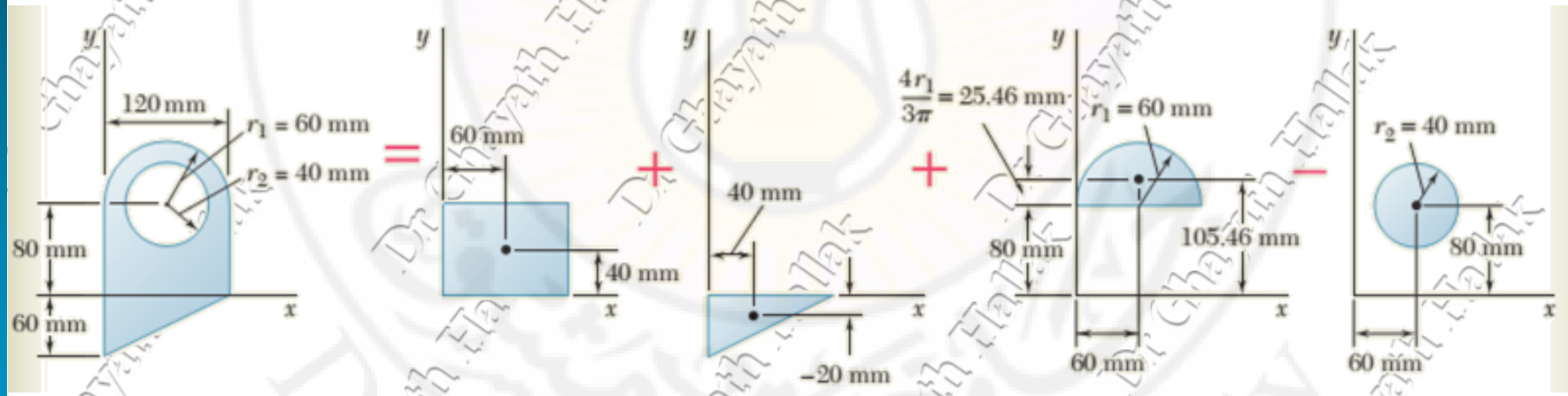
This approach will become clear by doing examples!

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EXAMPLE

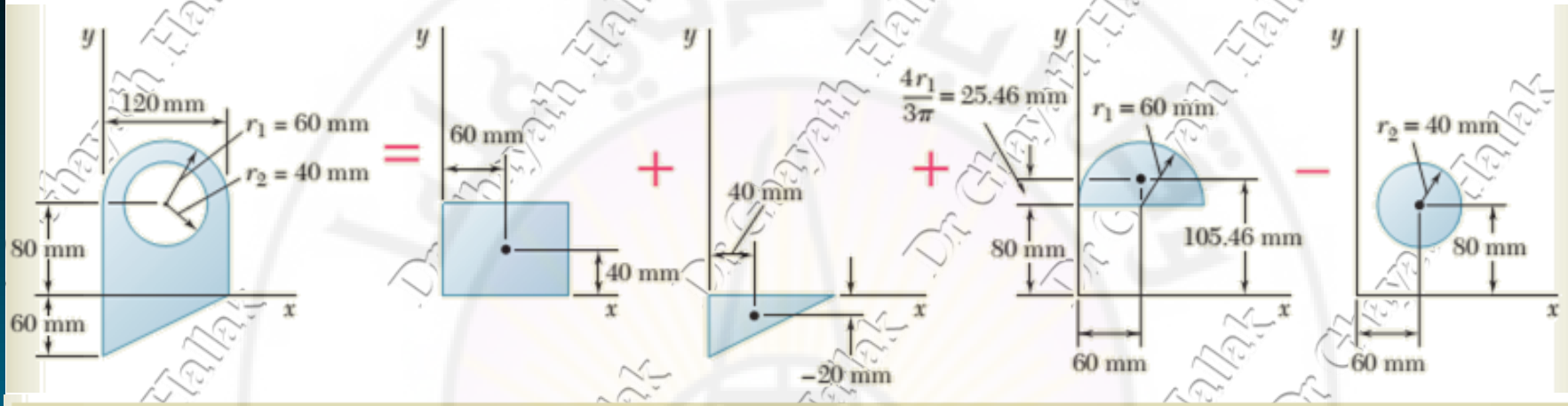
For the plane area shown, determine (*a*) the first moments with respect to the *x* and *y* axes, (*b*) the location of the centroid.





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EXAMPLE



Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma \bar{x}A = +757.7 \times 10^3$	$\Sigma \bar{y}A = +506.2 \times 10^3$

a. First Moments of the Area. $Q_x = \Sigma \bar{y}A = 506.2 \times 10^3 \text{ mm}^3$

$Q_y = \Sigma \bar{x}A = 757.7 \times 10^3 \text{ mm}^3$

b. Location of Centroid.

$\bar{X}\Sigma A = \Sigma \bar{x}A$: $\bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$

$\bar{X} = 54.8 \text{ mm}$

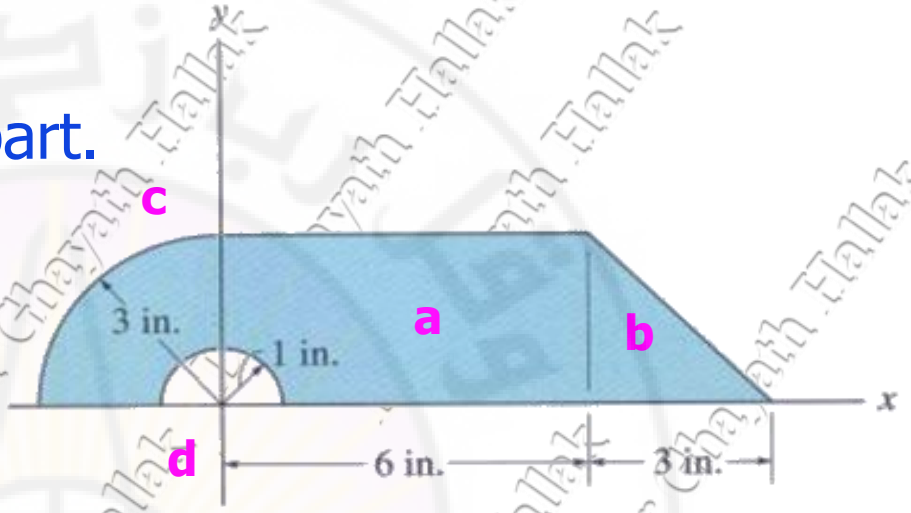
$\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$

$\bar{Y} = 36.6 \text{ mm}$

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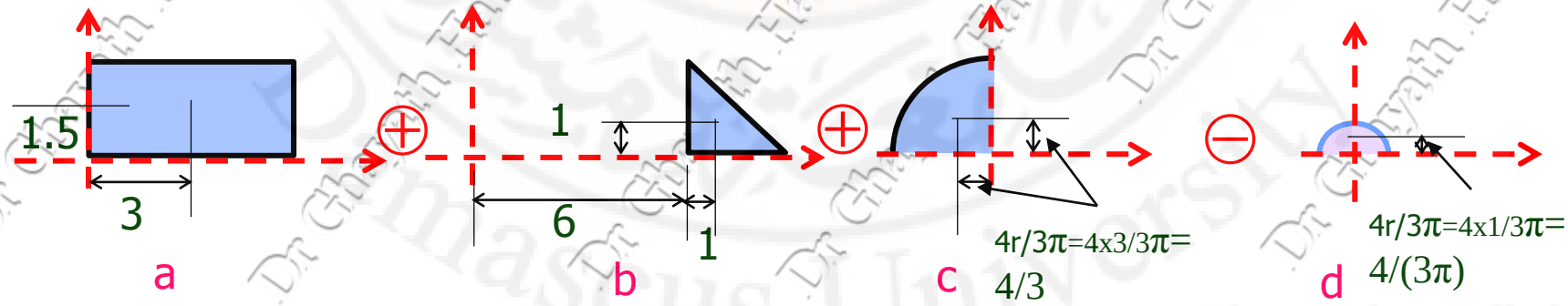
EXAMPLE

Find: The centroid of the part.

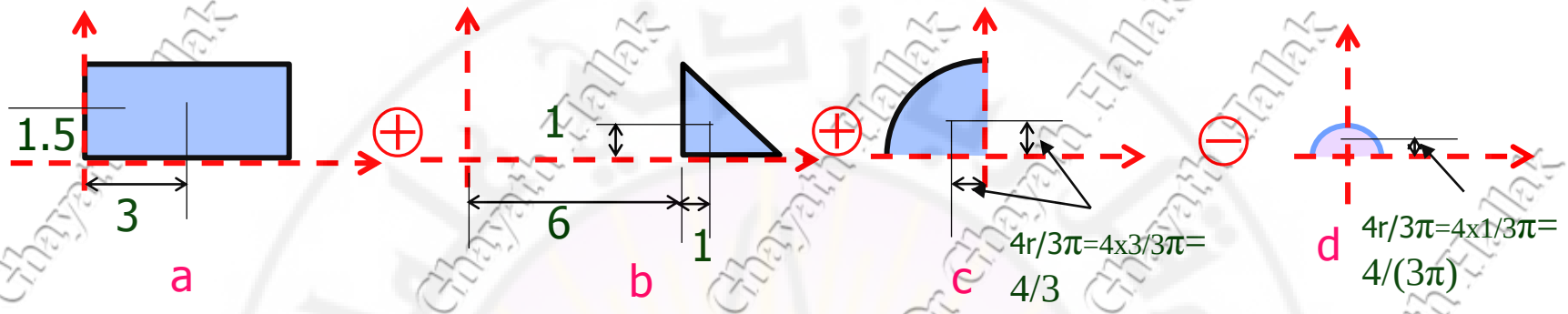


Solution:

1. This body can be divided into the following pieces: rectangle (a) + triangle (b) + quarter circular (c) – semicircular area (d)



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Segment	Area A (in^2)	$\tilde{x}$ (in)	$\tilde{y}$ (in)	$A\tilde{x}$ (in^3)	$A\tilde{y}$ (in^3)
Rectangle	18	3	1.5	54	27
Triangle	4.5	7	1	31.5	4.5
Q. Circle	$9\pi/4$	$-4(3)/(3\pi)$	$4(3)/(3\pi)$	−9	9
Semi-Circle	$-\pi/2$	0	$4(1)/(3\pi)$	0	-2/3
Σ	28.0			76.5	39.83

$$x_c = (\Sigma \tilde{x} A) / (\Sigma A) = 76.5\ in^3 / 28.0\ in^2 = 2.73\ in$$

$$y_c = (\Sigma \tilde{y} A) / (\Sigma A) = 39.83\ in^3 / 28.0\ in^2 = 1.42\ in$$

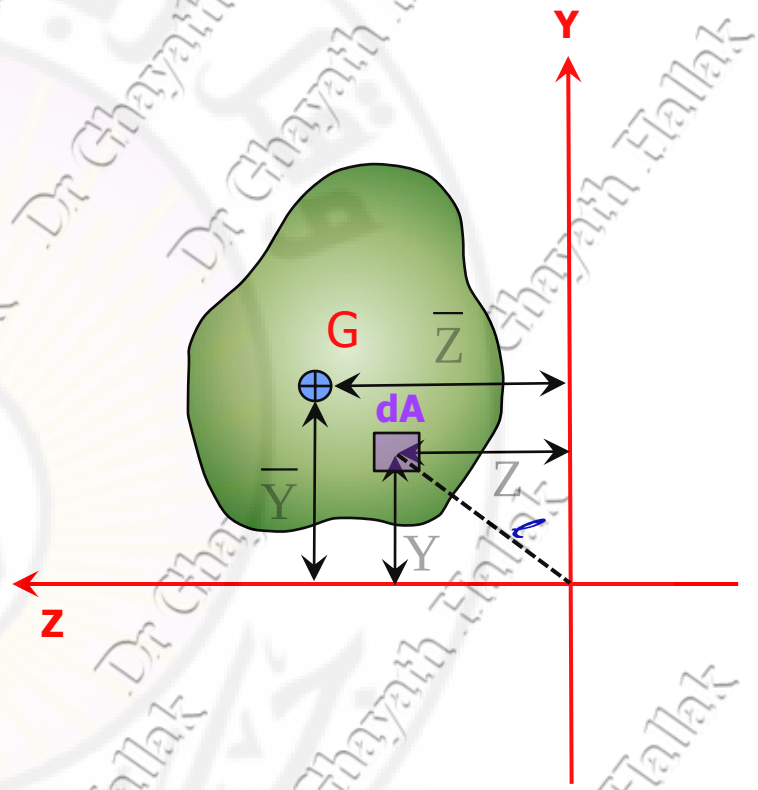
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MOMENT OF INERTIA
The Second moment of area

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1- The Second moment of area, The MOMENT OF INERTIA: (mm^4, m^4)

The **Moment of Inertia (I)** is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as Z or Y. It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis. The reference axis is usually a centroidal axis (NOT "Y & Z" axes shown in the Fig). The moment of Inertia expressed mathematically as:

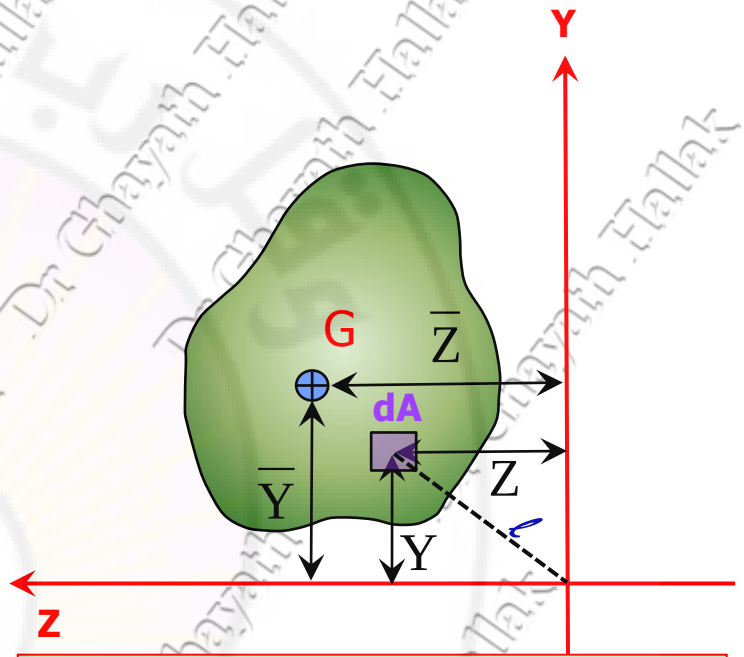


$$I_Z = \int_A Y^2\, dA \;,\; I_Y = \int_A Z^2\, dA$$

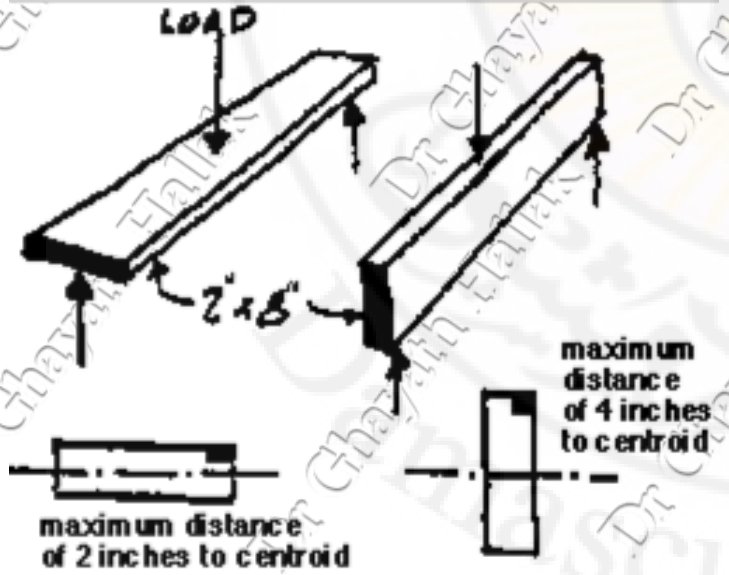
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1- The Second moment of area, The MOMENT OF INERTIA: (mm^4, m^4)

The Moment of Inertia is an important value which is used to determine the state of stress in a section, to calculate the resistance to buckling, and to determine the amount of deflection in a beam.



$$I_Z = \int_A Y^2\, dA \;,\; I_Y = \int_A Z^2\, dA$$

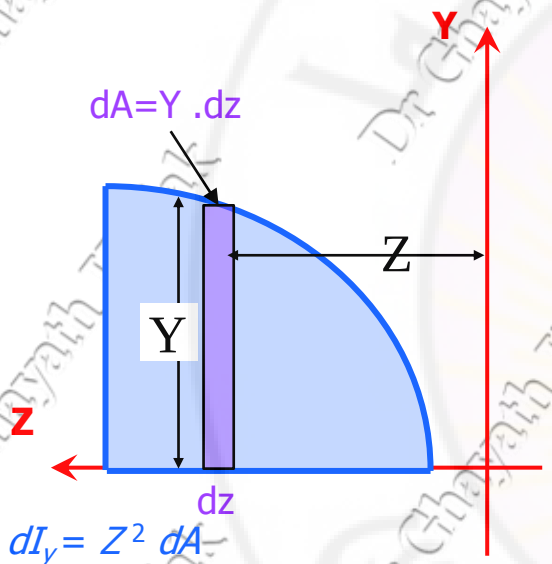


Both boards have the same cross-sectional area, but the area is distributed differently about the horizontal centroidal axis.

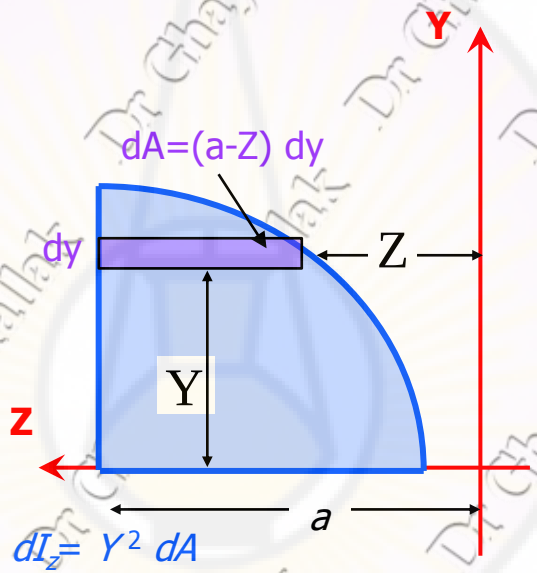
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1- The Second moment of area, The MOMENT OF INERTIA: (mm^4, m^4)

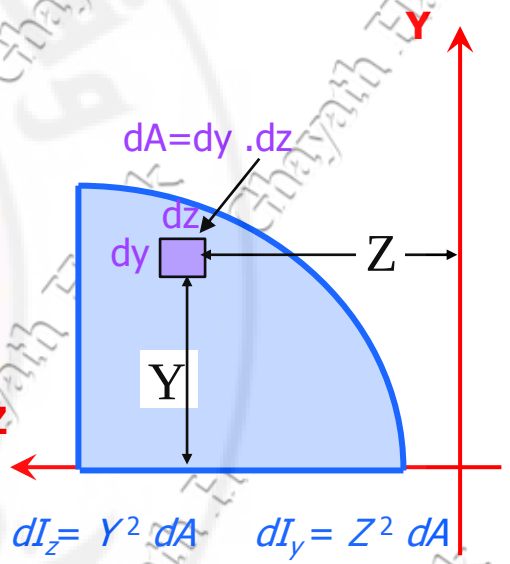
DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION



$dI_y = Z^2\, dA$



$dI_z = Y^2\, dA$



$dI_z = Y^2\, dA$ $dI_y = Z^2\, dA$

The moment of inertia of an area is always positive

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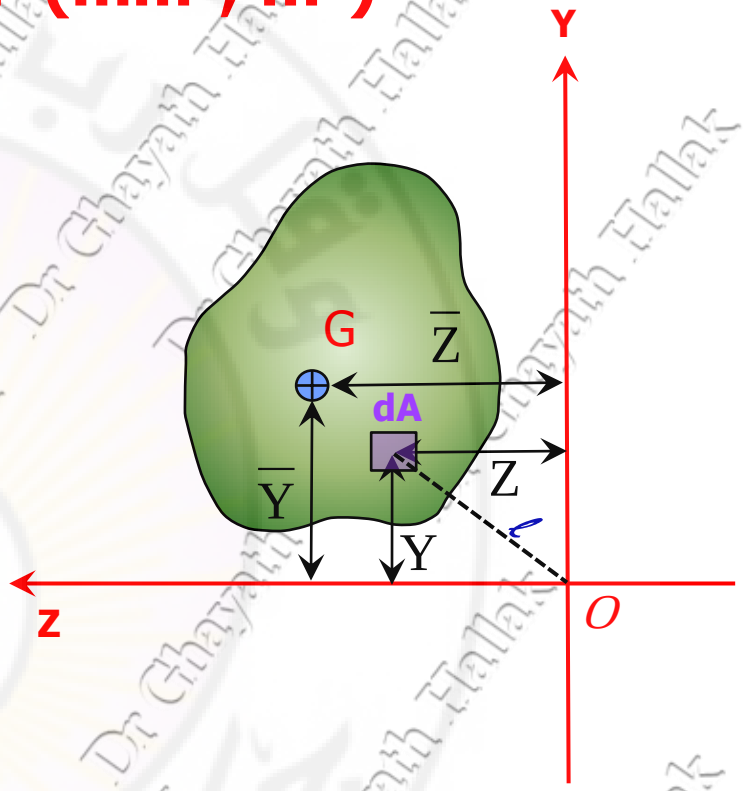
2- The polar moment of Inertia (mm^4, m^4)

Polar moment of inertia, denoted by J_O or I_p , is the area moment of inertia about the *X*-axis (perpendicular to plan of cross-section area) given by:

$$I_p = J_o = \int_A l^2\, dA = \int_A (Z^2 + Y^2)\, dA$$

$$= \int_A Z^2\, dA + \int_A Y^2\, dA$$

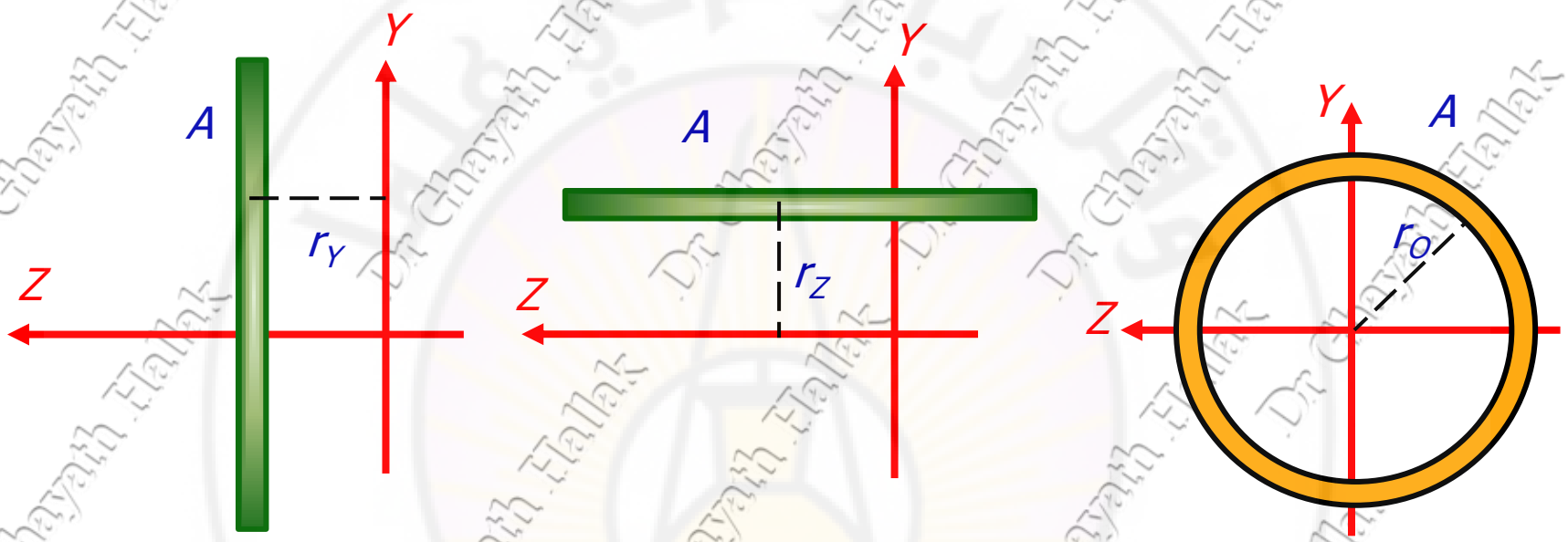
$$I_p = J_o = I_Y + I_Z$$



This integral of great importance in problems concerning the torsion of cylindrical shafts and in problems dealing with the rotation of slabs

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3- Radius of gyration (mm^3, m^3)

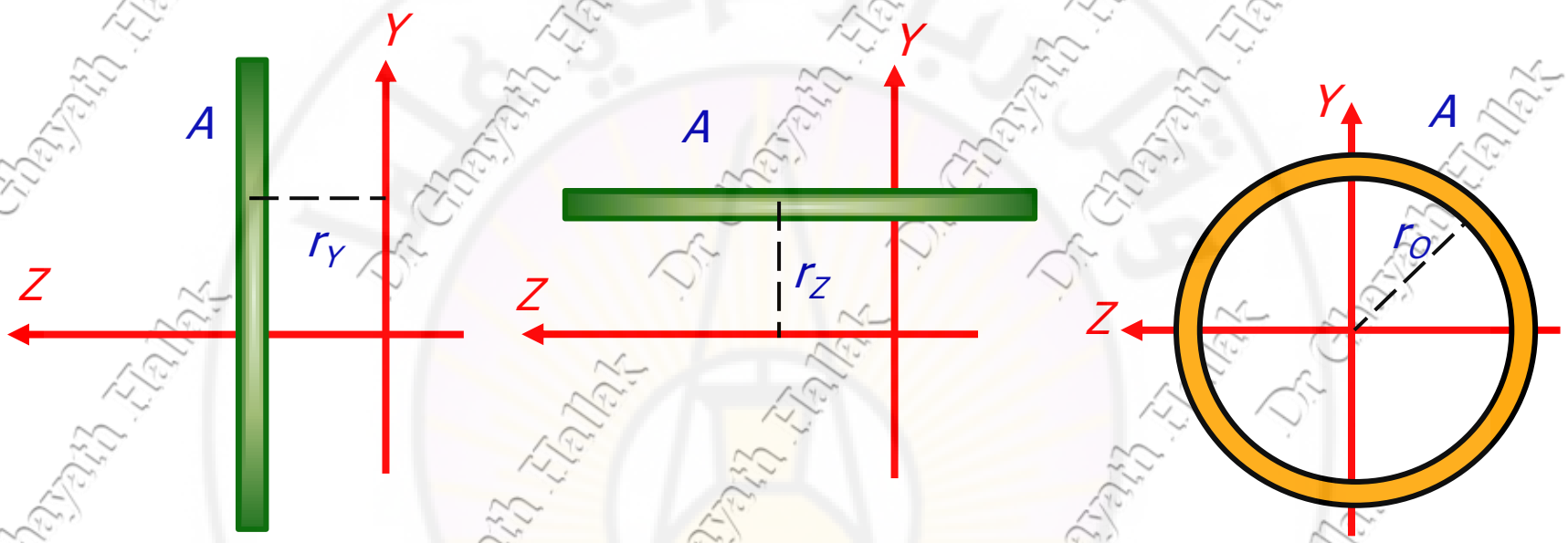


The radius of gyration is the distance *r* away from the axis that all the area can be concentrated to result in the same moment of inertia. That is,

$$I = r^2 A = i^2 A$$

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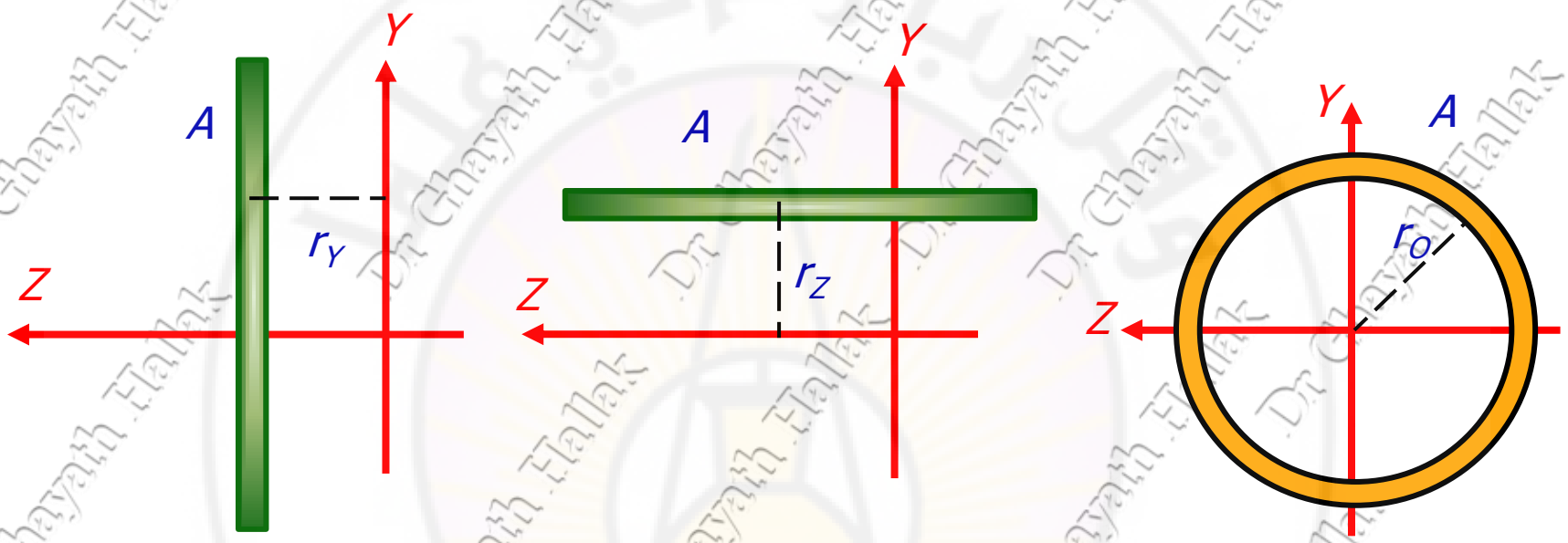
3- Radius of gyration (mm^3, m^3)



For a given area, one can define the radius of gyration around the *Y*-axis, denoted by r_Y ,(i_Y) the radius of gyration around the *Z*-axis, denoted by r_Z , (i_Z) and the radius of gyration around the *X*-axis, denoted by r_O, (i_O). These are calculated from the relations:

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3- Radius of gyration (mm³, m³)

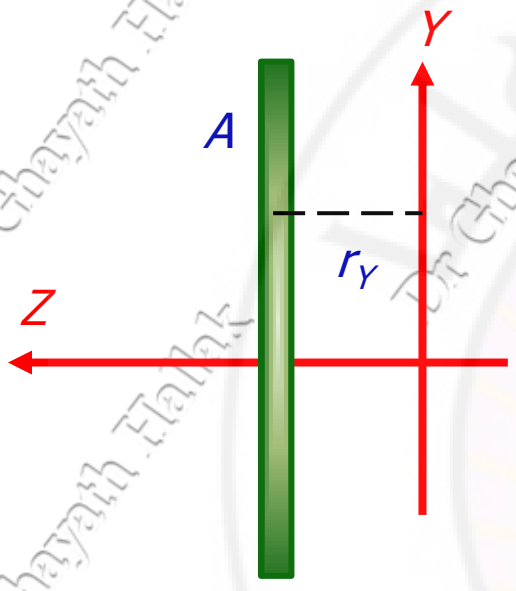


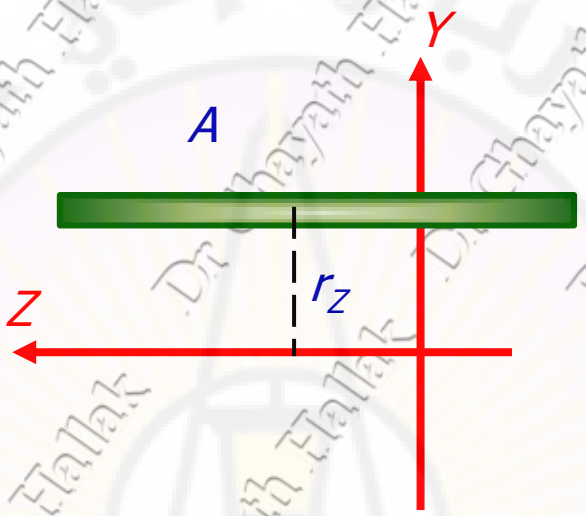
$$i_Z^2 = r_Z^2 = \frac{I_Z}{A}, \quad i_Y^2 = r_Y^2 = \frac{I_Y}{A}, \quad i_O^2 = r_O^2 = \frac{J_O}{A}$$

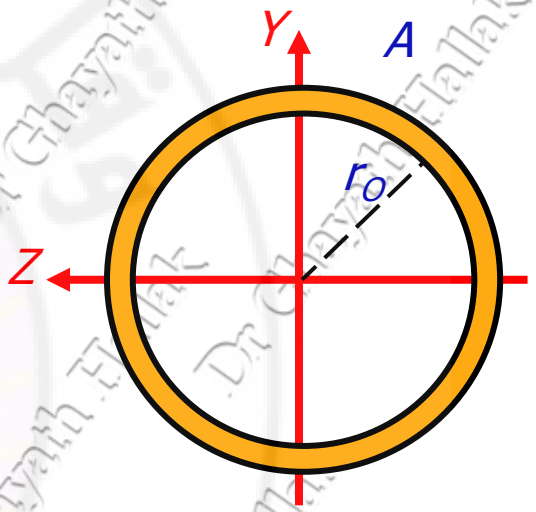
$$i_Z = r_Z = \sqrt{\frac{I_Z}{A}}, \quad i_Y = r_Y = \sqrt{\frac{I_Y}{A}}, \quad i_O = r_O = \sqrt{\frac{J_O}{A}}$$

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3- Radius of gyration (mm^3, m^3)







It can easily to show from $J_O = I_Y + I_Z$ that

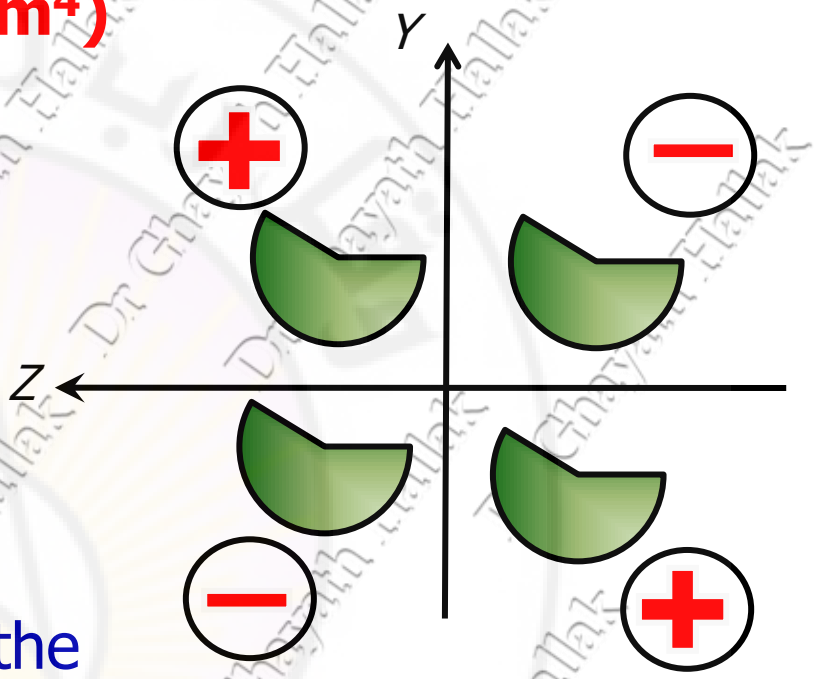
$$r_Z^2 + r_Y^2 = r_O^2$$

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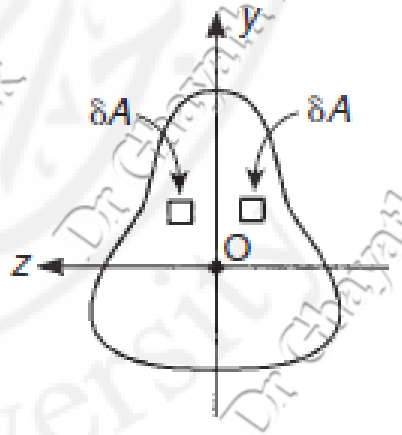
4- Product of Inertia (mm^4, m^4)

The product second moment of area, I_{zy} , of a beam section with respect to z and y axes is defined by:

$$I_{ZY} = \int_A Z\, Y\, dA$$



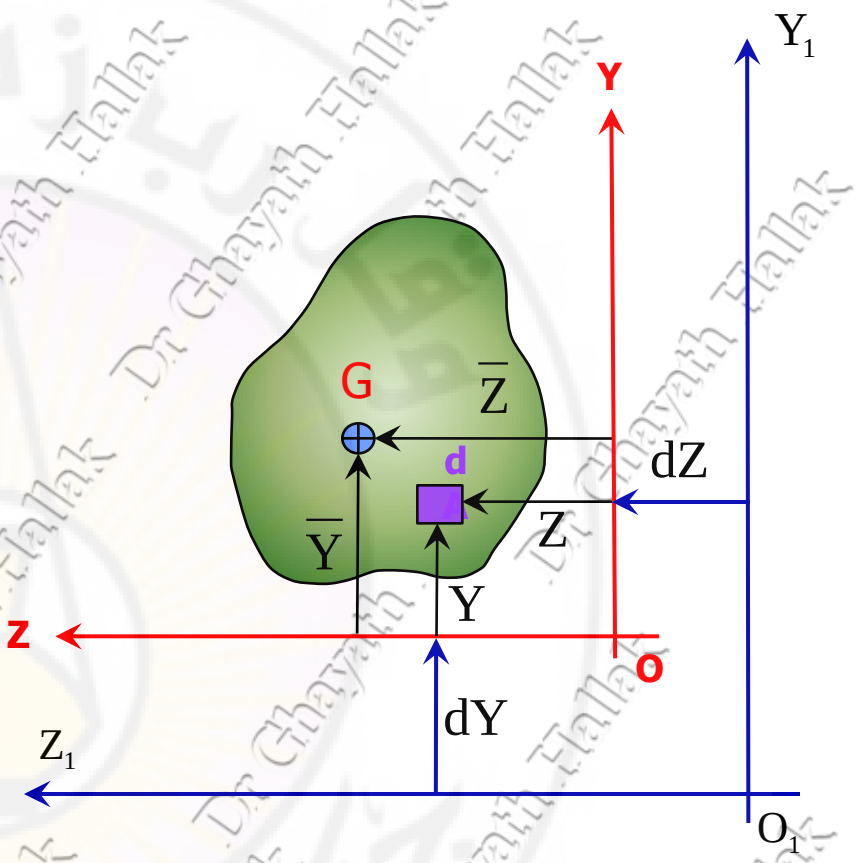
when one (or both) of the coordinate axes is an axis of symmetry $\Rightarrow$ $I_{zy} = 0$



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5- Parallel-Axis Theorems:

Suppose that we know the value of I_Y, I_Z and I_{ZY}. We need to determine the value of I_{Y1}, I_{Z1} and I_{Y1Z1} (moment of Inertia according to the new axes Y1 and Z1 "parallel axis")



$$I_{Z1} = \int_A Y_1^2 dA$$

$$I_{Z1} = \int_A (Y + dY)^2 dA$$

$$I_{Z1} = \int_A Y^2 dA + dY^2 \int_A dA + 2\, dY \int_A Y\, dA$$

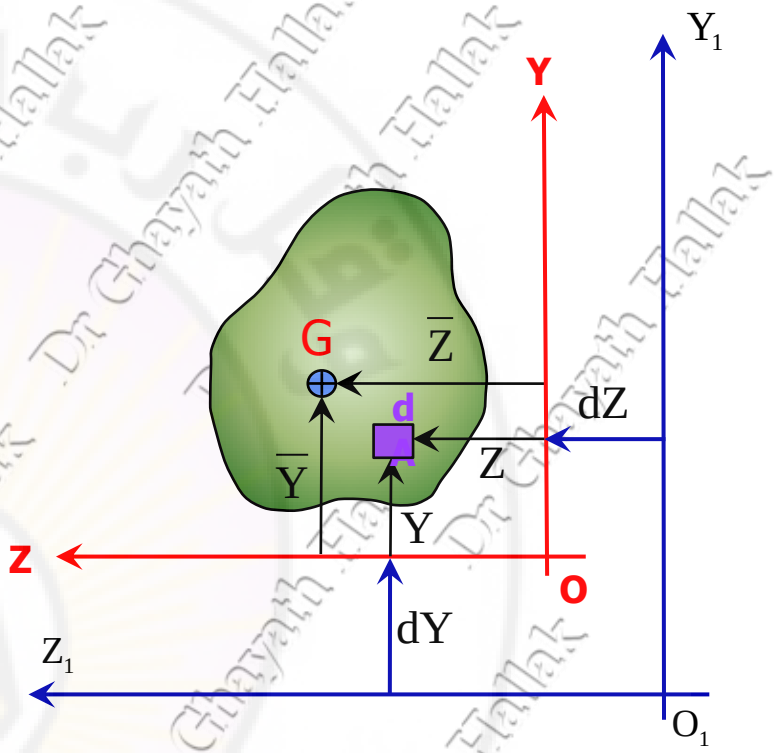
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5- Parallel-Axis Theorems:

$$I_{Z1} = I_Z + 2\ dY\ Q_Z + dY^2\ A$$

Similarly

$$I_{Y1} = I_Y + 2\ dZ\ Q_Y + dZ^2\ A$$



Product of Inertia

$$I_{Z1Y1} = \int_A Y_1\ Z_1\ dA = \int_A (Y + dY)(Z + dZ)\ dA$$

$$I_{Z1Y1} = \int_A Y\ Z\ dA + dY \int_A Z\ dA + dZ \int_A Y\ dA + dY\ dZ \int_A dA$$

$$I_{Z1Y1} = I_{ZY} + dY\ Q_Y + dZ\ Q_Z + dY\ dZ\ A$$

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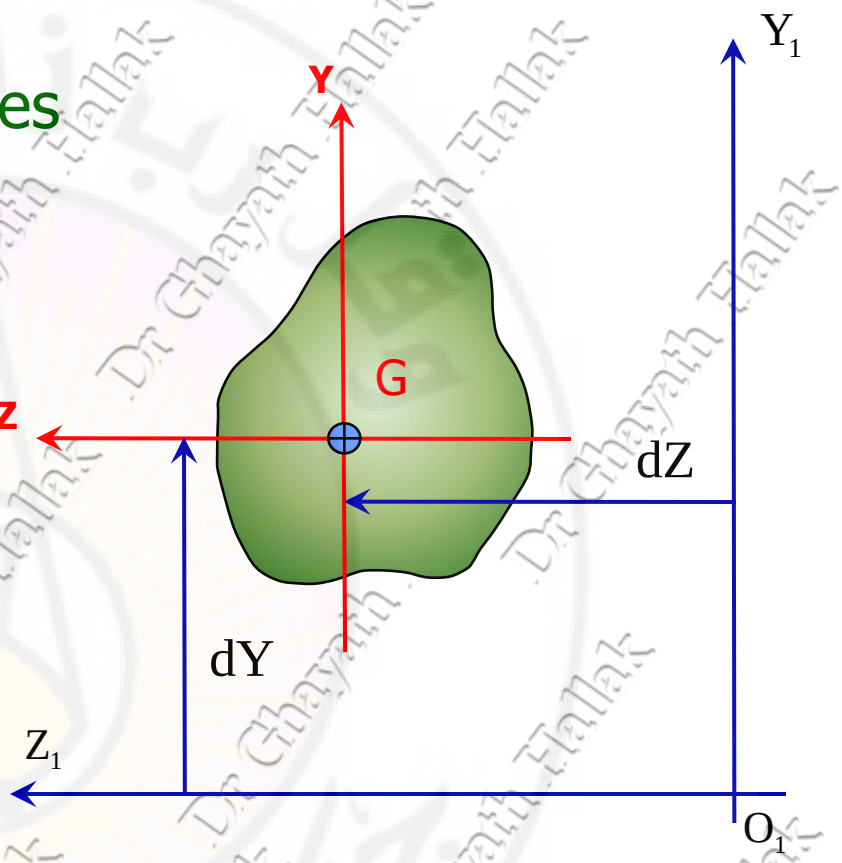
5- Parallel-Axis Theorems:

IF Z & Y are CENTROIDAL axes Then

$$I_{Z1} = I_{GZ} + dY^2\ A$$

$$I_{Y1} = I_{GY} + dZ^2\ A$$

$$I_{Z1Y1} = I_{GZY} + dY\ dZ\ A$$

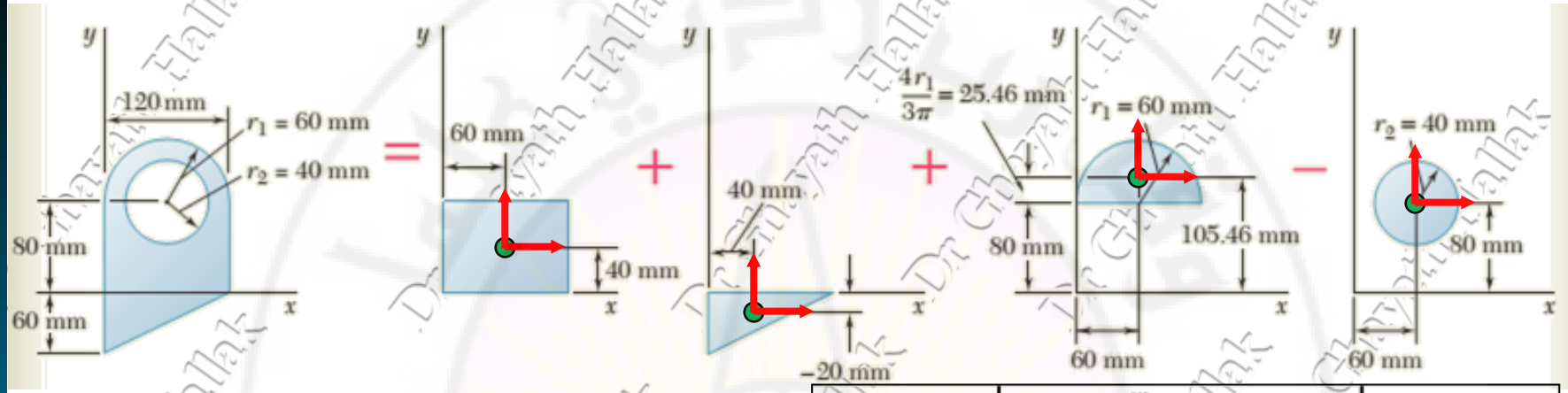


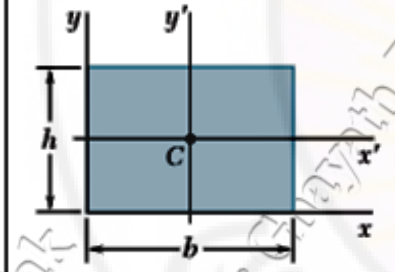
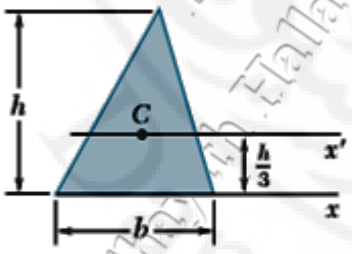
It can be seen from Eqs. above that if either G*Z or GY is an axis of symmetry, i.e.* I_{GZY} *=0, then* $I_{Z1Y1} = dY\ dZ\ A$

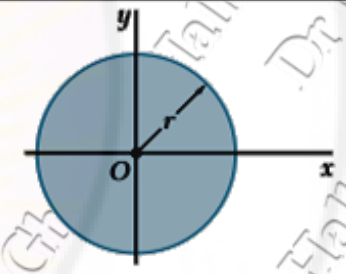
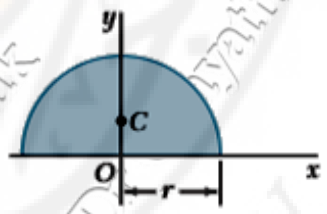
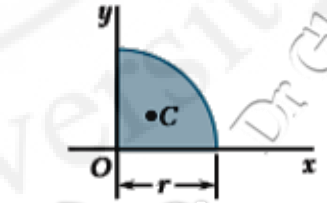
Thus for a section component having an axis of symmetry that is parallel to either of the section reference axes the product second moment of area is the product of the coordinates of its centroid multiplied by its area.

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5- MOMENTS OF INERTIA OF COMPOSITE AREAS



Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$

Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$

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EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Determine the moment of inertia for the rectangular area with respect to (a) the centroidal x′ axis, (b) the centroidal x′ axis, (c) the axis x_b passing through the base of the rectangular, and (d) the pole or z′ axis perpendicular to the x′-y′ plane and passing through the centroid C.

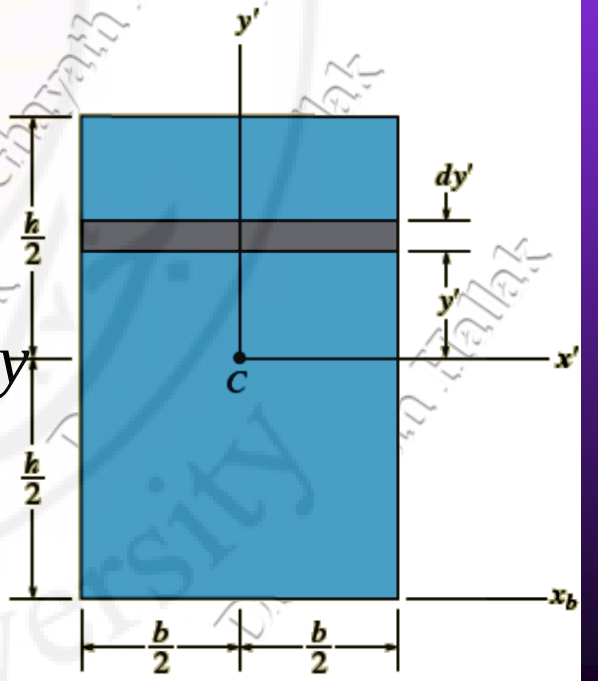
Part (a)

Differential element chosen, distance y′ from x′ axis.

Since dA = b dy′,

$$\bar{I}_x = \int_A y'^2\, dA = \int_{-h/2}^{h/2} y'^2 (b\,dy') = b\int_{-h/2}^{h/2} y'^2\, dy'$$

$$\bar{I}_x = b\left[\frac{y'^3}{3}\right]_{-h/2}^{+h/2} = \frac{1}{12}bh^3$$



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EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

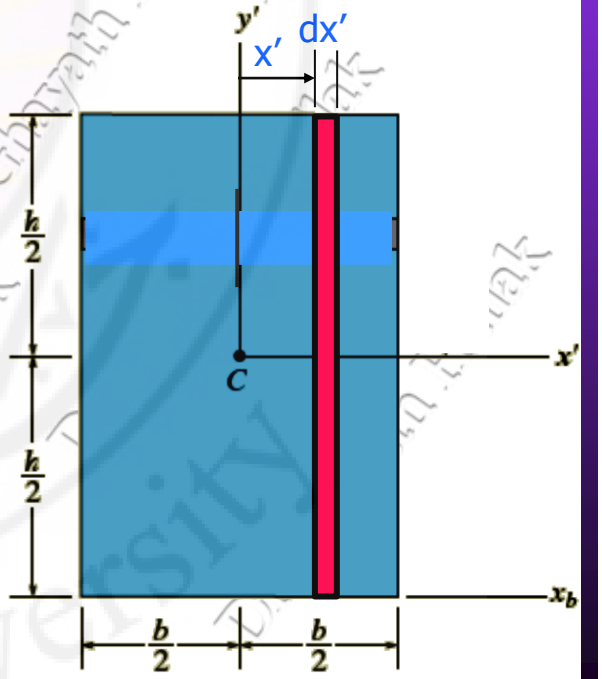
Part (b)

Differential element chosen, distance x′ from y′ axis.

Since dA = h dx′,

$$\bar{I}_y = \int_A x'^2\, dA = \int_{-b/2}^{b/2} x'^2\,(h\,dx') = h\int_{-b/2}^{b/2} x'^2\, dy$$

$$\bar{I}_y = h\left[\frac{x'^3}{3}\right]_{-b/2}^{+b/2} = \frac{1}{12}hb^3$$



Part (c)

By applying parallel axis theorem,

$$I_{x_b} = \bar{I}_x + Ad^2 = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$

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EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Part (d)

For polar moment of inertia about point C,

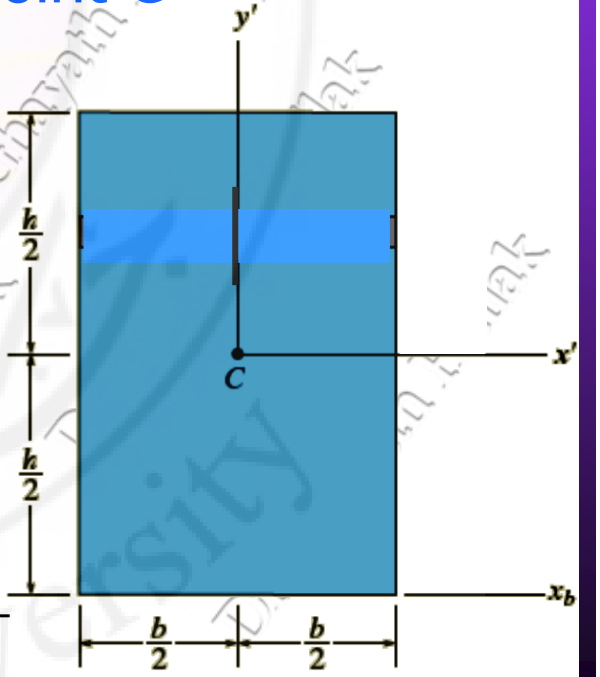
$$J_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12} bh(h^2 + b^2)$$

For radius of gyration about axis x', y' & point C

$$r_{x'} = \sqrt{\frac{\bar{I}_{x'}}{A}} = \sqrt{\frac{bh^3/12}{bh}} = \frac{h}{2\sqrt{3}}$$

$$r_{y'} = \sqrt{\frac{\bar{I}_{y'}}{A}} = \sqrt{\frac{hb^3/12}{bh}} = \frac{b}{2\sqrt{3}}$$

$$r_C^{\,2} = r_{z'}^{\,2} = r_{x'}^{\,2} + r_{y'}^{\,2} = \frac{h^2}{12} + \frac{b^2}{12} = \frac{h^2 + b^2}{12}$$

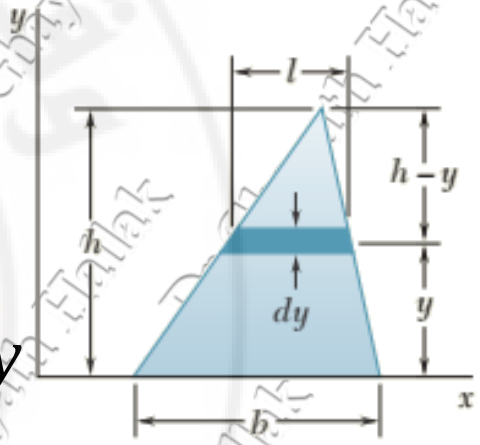


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EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Determine the moment of inertia of a triangle with respect to its base.

$dI_x = y^2\, dA\,,\; dA = \ell\, dy$



Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \Rightarrow l = b\frac{h-y}{h}, \therefore dA = b\frac{h-y}{h}dy$$

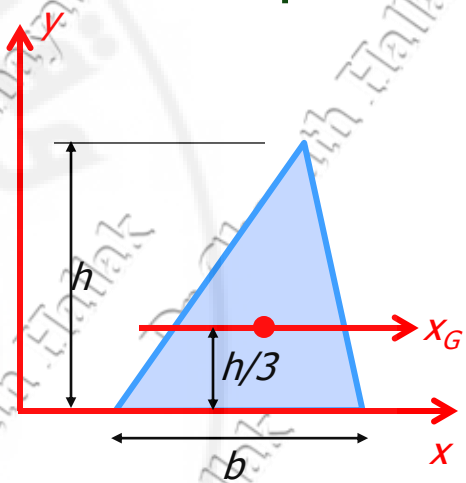
$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$I_x = \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h = \frac{bh^3}{12}$$

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EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Determine the moment of inertia for the area with respect to x_G axis



By applying parallel axis theorem,

$$I_x = \bar{I}_{xG} + Ad^2 \Rightarrow \bar{I}_{xG} = I_x - Ad^2$$

$$\bar{I}_{xG} = \frac{1}{12}bh^3 - \frac{bh}{2}\left(\frac{h}{3}\right)^2 = \frac{1}{36}bh^3$$

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EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

(a) Determine the moment of inertia of a circular area with respect to a diameter.

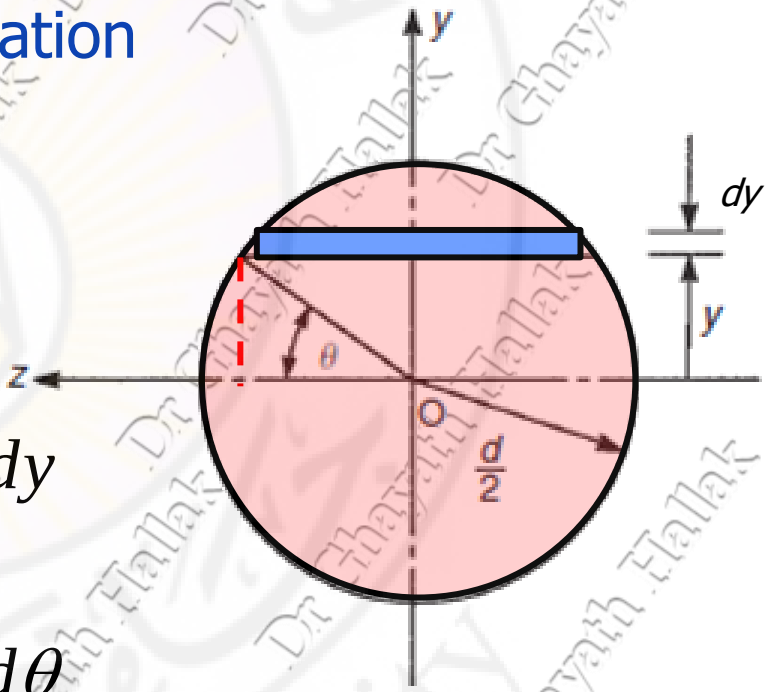
(b) Determine the centroidal polar moment of inertia of a circular area by direct integration

$$dA = 2\left(\frac{d}{2}\cos\theta\right)dy$$

$$Iz = \int_A y^2 dA = \int_{-d/2}^{d/2} 2\left(\frac{d}{2}\cos\theta\right) y^2 dy$$

$$y = \left(\frac{d}{2}\sin\theta\right) \Rightarrow dy = \left(\frac{d}{2}\cos\theta\right)d\theta$$

$$y = \mp d/2 \Rightarrow \theta = \mp\pi/2$$



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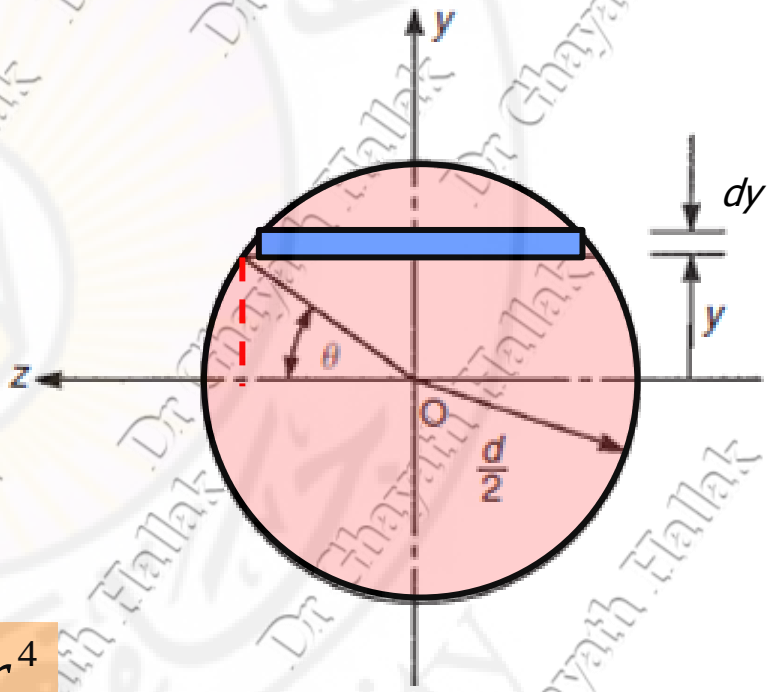
EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

$$Iz = \int_{-\pi/2}^{\pi/2} (d\cos\theta)\left(\frac{d}{2}\sin\theta\right)^2\left(\frac{d}{2}\cos\theta\right)d\theta$$

$$Iz = \frac{d^4}{8}\int_{-\pi/2}^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta$$

$$\Rightarrow Iz = \frac{\pi d^4}{64} = Iy$$

$$\Rightarrow Iz = \frac{\pi r^4}{4} = Iy$$

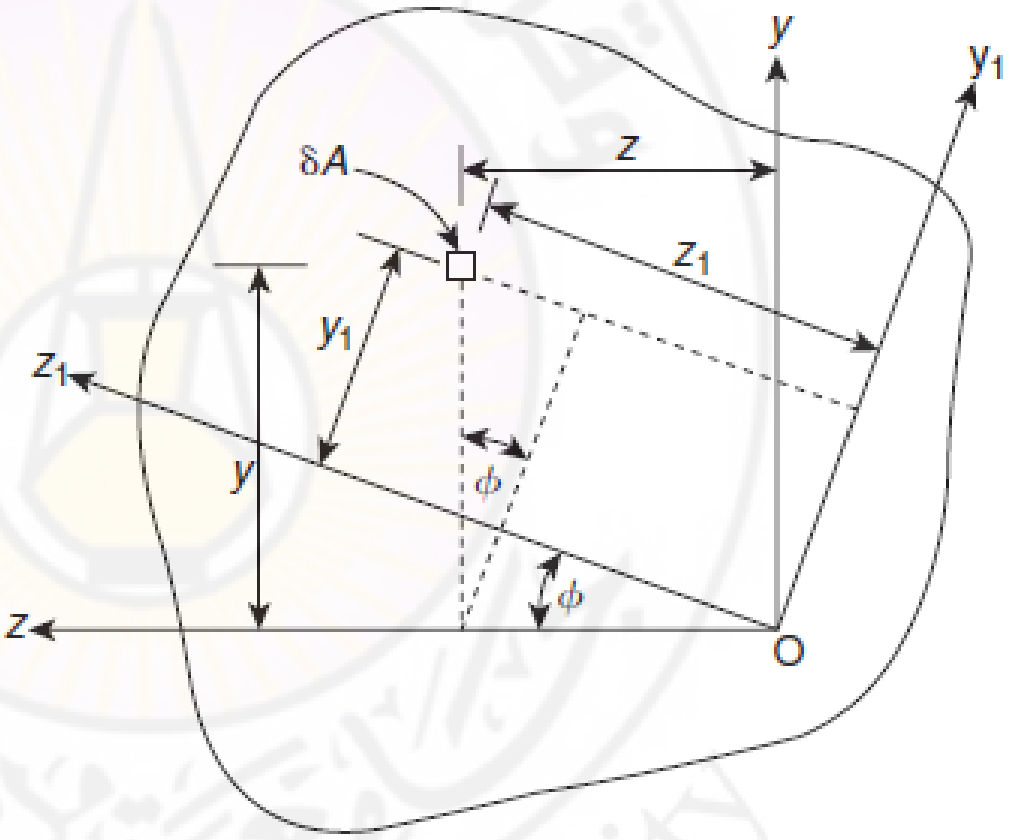


$$Jo = Iz + Iy = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$

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PRINCIPAL AXES AND PRINCIPAL SECOND MOMENTS OF AREA

In any beam section there is a set of axes, neither of which need necessarily be an axis of symmetry, for which **the product second moment of area is zero**. Such axes are known as ***principal axes*** *and the second moments of area about these axes are termed* **principal second moments of area**.



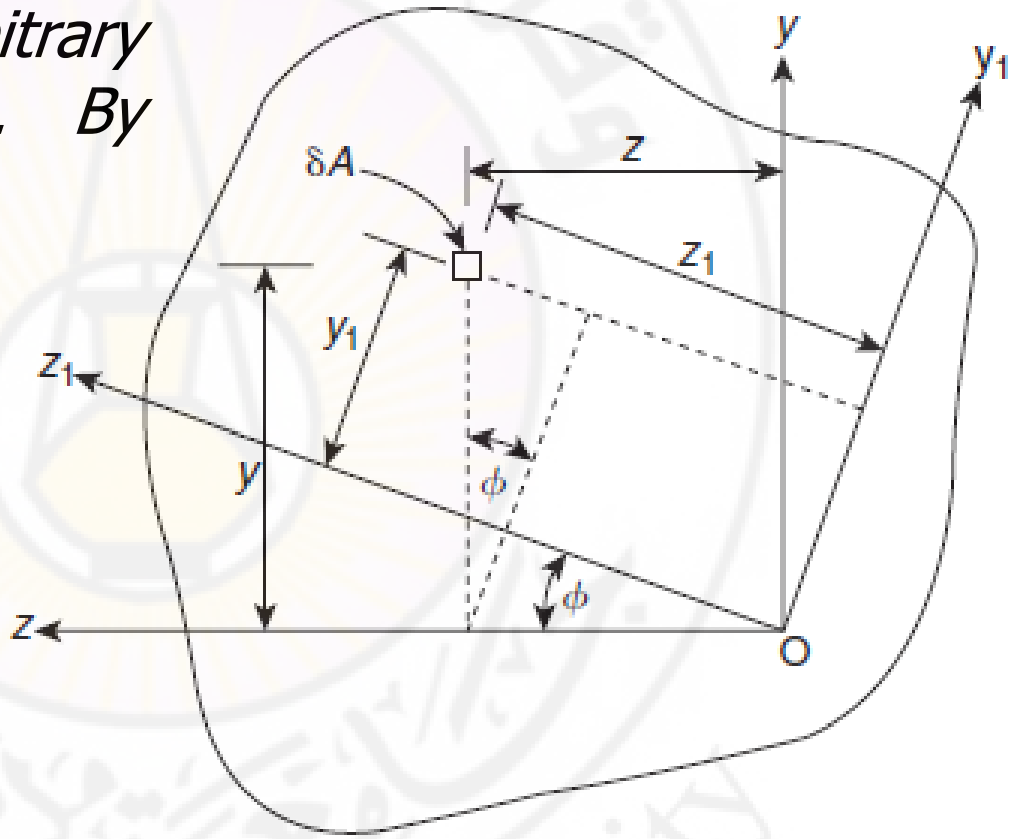
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Suppose that the second moments of area *Iz, Iy and the product second moment of area, Izy, about arbitrary* axes O*zy are known. By definition*

$$I_y = \int_A z^2 \, dA$$

$$I_z = \int_A y^2 \, dA$$

$$I_{zy} = \int_A y \, z \, dA$$



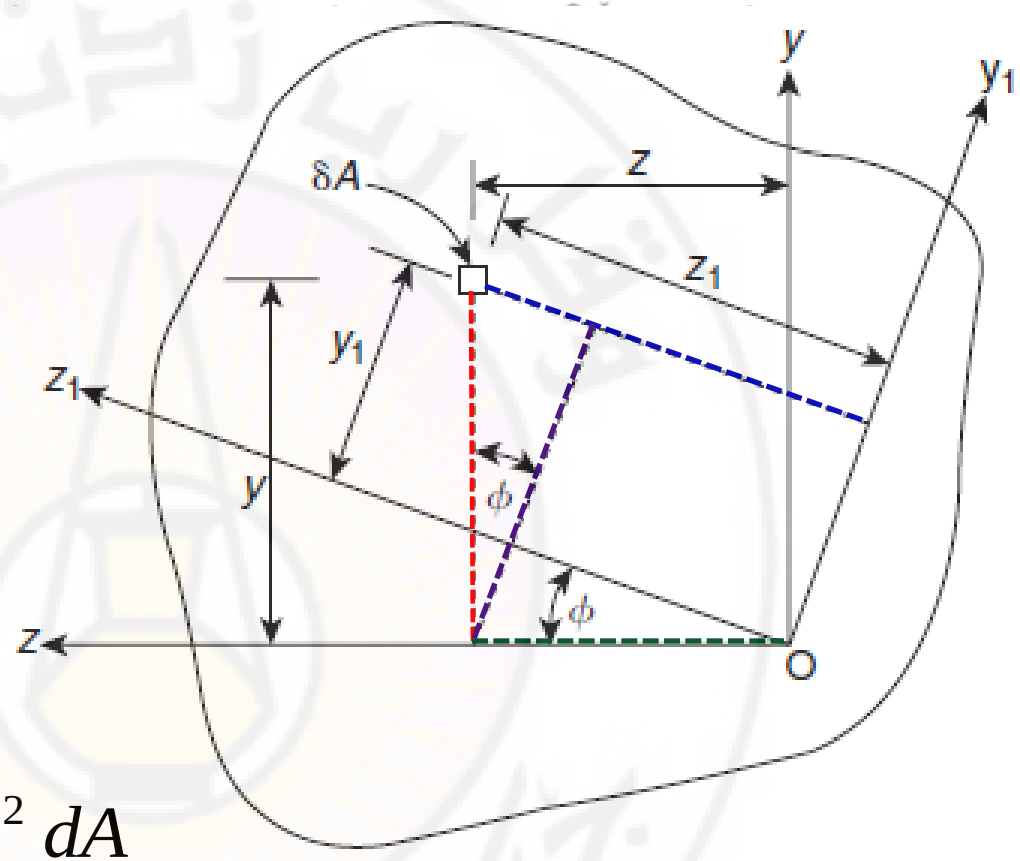
The corresponding second moments of area about axes Oz_1y_1 *are*

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$$I_{y1} = \int_A z_1^{\,2}\, dA$$

$$I_{z1} = \int_A y_1^{\,2}\, dA$$

$$I_{z1y1} = \int_A y_1\, z_1\, dA$$



From Fig.

$z_1 = z \cos\varphi + y \sin\varphi$

$y_1 = y \cos\varphi - z \sin\varphi$

$$I_{y1} = \int_A (z\cos\phi + y\sin\phi)^2\, dA$$

$$I_{y1} = \int_A z^2 \cos^2\phi\, dA + \int_A y^2 \sin^2\phi\, dA + 2\int_A y\, z \sin\phi\cos\phi\, dA$$

$$I_{y1} = I_y \cos^2\phi + I_z \sin^2\phi + I_{zy} \sin 2\phi \qquad (1)$$

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Similarly, we find

$$I_{z1} = I_z \cos^2\phi + I_y \sin^2\phi - I_{zy} \sin 2\phi \qquad (2)$$

$$I_{z1y1} = \left(\frac{I_z - I_y}{2}\right) \sin 2\phi + I_{zy} \cos 2\phi \qquad (3)$$

Equations (1, 2 &3) give the second moments of area and product second moment of area about axes inclined at an angle φ *to the z axis. In the special case where* Oz_1y_1 are principal axes, $I_{z(p),y(p)} = 0$, $\varphi = \varphi_p$ *and* Eqs. *(1) and (2) become*

$$I_{yp} = I_y \cos^2\phi_p + I_z \sin^2\phi_p + I_{zy} \sin 2\phi_p \qquad (4)$$

$$I_{zp} = I_z \cos^2\phi_p + I_y \sin^2\phi_p - I_{zy} \sin 2\phi_p \qquad (5)$$

Since $I_{z1y1} = I_{zp,yp} = 0$, *Eq. (3) gives*

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$$0 = \left(\frac{I_z - I_y}{2} \right) \sin 2\phi_p + I_{zy} \cos 2\phi_p \Rightarrow$$

$$\tan 2\phi_p = \frac{2I_{zy}}{I_y - I_z} \qquad \textbf{(6)}$$

The angle φ_p *may be eliminated from Eqs (4) and (5) by first determining* cos $2\varphi_p$ *and* sin $2\varphi_p$ *using Eq. (6). We have from trigonometry (double-angle formulas)*

$$\sin\alpha \cdot \cos\alpha = \frac{1}{2}\sin 2\alpha, \quad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha), \quad \sin 2\alpha = \frac{\tan 2\alpha}{\sqrt{1+\tan^2 2\alpha}}$$

$$\cos 2\alpha = \frac{1}{\sqrt{1+\tan^2 2\alpha}}$$

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Thus

$$\sin 2\phi_p = \frac{\tan 2\phi_p}{\sqrt{1+\tan^2 2\phi_p}} = \frac{2I_{zy}/(I_y - I_z)}{\sqrt{1+4I_{zy}^2/(I_y - I_z)^2}} = \frac{I_{zy}}{\sqrt{[(I_y - I_z)/2]^2 + I_{zy}^2}}$$

$$\cos 2\alpha = \frac{1}{\sqrt{1+\tan^2 2\alpha}} = \frac{1}{\sqrt{1+4I_{zy}^2/(I_y - I_z)^2}} = \frac{(I_y - I_z)/2}{\sqrt{[(I_y - I_z)/2]^2 + I_{zy}^2}}$$

Rewriting Eq. (5) in terms of $\cos 2\varphi_p$ *and* $\sin 2\varphi_p$ *we have*

$$I_{zp} = \frac{I_z}{2}(1+\cos 2\phi_p) + \frac{I_y}{2}(1-\cos 2\phi_p) - I_{zy}\sin 2\phi_p$$

Substituting for $\cos 2\varphi_p$ *and* $\sin 2\varphi_p$ *from the above we obtain*

$$I_{zp} = \frac{I_z + I_y}{2} - \frac{1}{2}\sqrt{(I_z - I_y)^2 + 4I^2{}_{zy}}$$

Similarly

$$I_{yp} = \frac{I_z + I_y}{2} + \frac{1}{2}\sqrt{(I_z - I_y)^2 + 4I^2{}_{zy}}$$

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Note that the solution of Eq. (6) gives two values for the inclination of the principal axes, φ_p *and* $\varphi_p + \pi/2$, *corresponding to the axes* Oz_p *and* Oy_p.

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MOMENTS OF INERTIA OF COMPOSITE AREAS

Determine the product of inertia of the right triangle shown (*a*) with respect to the *x* and *y* axes and (*b*) with respect to centroidal axes parallel to the *x* and *y* axes.

$$I_{xy} = \int \bar{x}.\bar{y}.dA$$

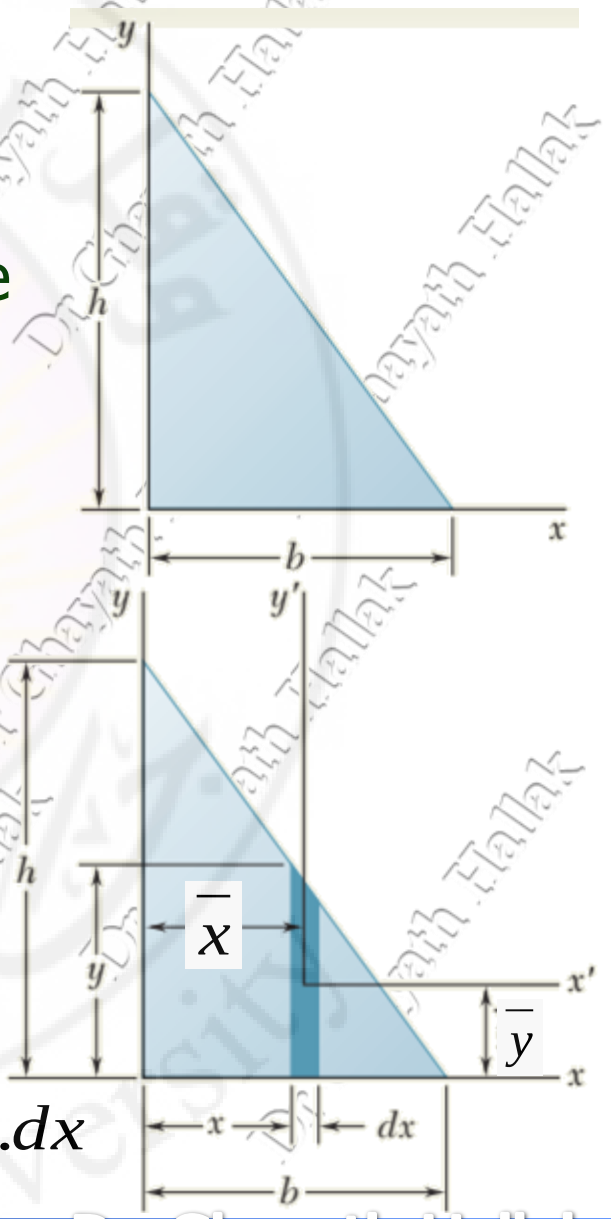
$$dA = y.dx$$

$$\frac{y}{h} = \frac{b-x}{b} \Rightarrow y = (b-x).\frac{h}{b}$$

$$\therefore \quad dA = (b-x).\frac{h}{b}.dx$$

$$\bar{x} = x, \quad \bar{y} = \frac{y}{2} = (b-x).\frac{h}{2b}$$

$$I_{xy} = \int \bar{x}.\bar{y}.dA = \int_0^b x\left[\frac{h}{2b}(b-x)\right].\frac{h}{b}(b-x).dx$$

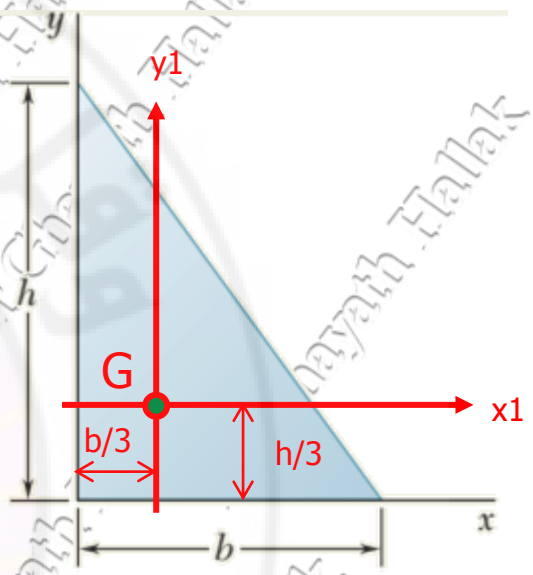


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MOMENTS OF INERTIA OF COMPOSITE AREAS

$$I_{xy} = \frac{h^2}{2b^2}\left[b^2\frac{x^2}{2} + \frac{x^4}{4} - 2b\frac{x^3}{3}\right]_0^b$$

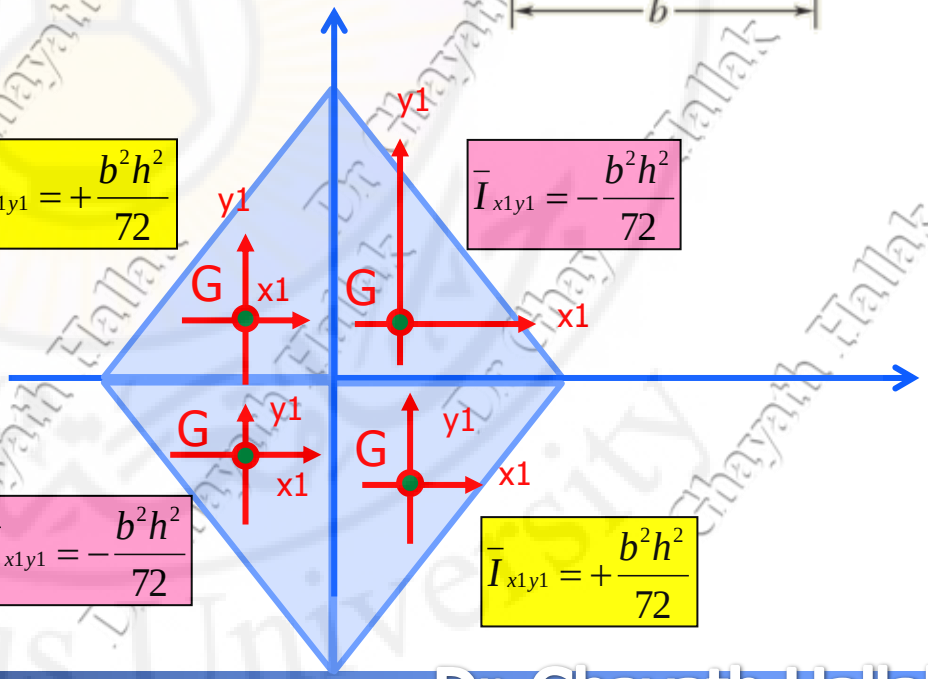
$$I_{xy} = \frac{b^2h^2}{24}$$



$$I_{XY} = \bar{I}_{x1y1} + (A_i)\times X_G \times Y_G \Rightarrow \bar{I}_{x1y1} = I_{XY} - (A_i)\times X_G \times Y_G =$$

$$\frac{b^2h^2}{24} - \left(\frac{b.h}{2}\right)\times\left(-\frac{b}{3}\right)\times\left(-\frac{h}{3}\right)$$

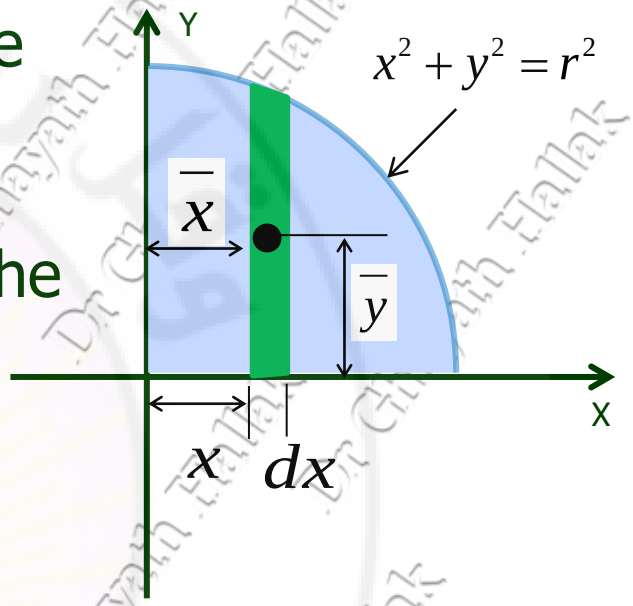
$$\bar{I}_{x1y1} = -\frac{b^2h^2}{72}$$



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MOMENTS OF INERTIA OF COMPOSITE AREAS

Determine the product of inertia of the Quarter circle shown (*a*) with respect to the *x* and *y* axes and (*b*) with respect to centroidal axes parallel to the *x* and *y* axes.



$$I_{xy} = \int \bar{x}.\bar{y}.dA$$

$$dA = y.dx$$

$$y^2 = r^2 - x^2 \Rightarrow y = \left(r^2 - x^2\right)^{0.5}$$

$$\therefore \quad dA = \left(r^2 - x^2\right)^{0.5}.dx$$

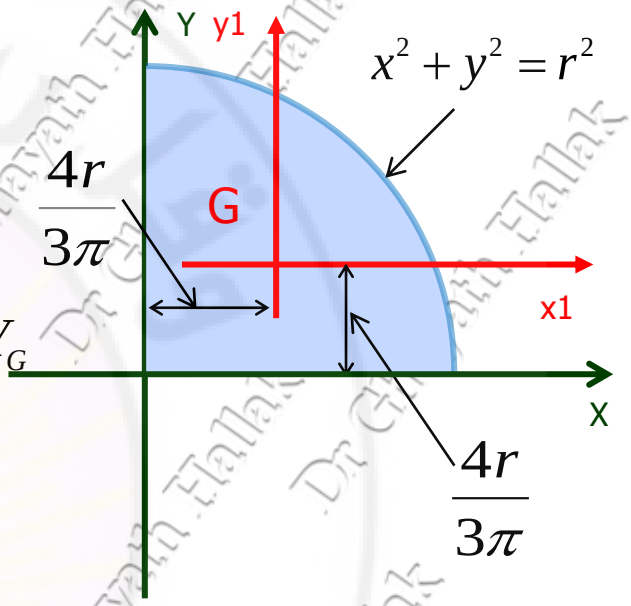
$$\bar{x} = x, \quad \bar{y} = \frac{y}{2} = \frac{\left(r^2 - x^2\right)^{0.5}}{2}$$

$$I_{xy} = \int_0^r x \frac{\left(r^2 - x^2\right)^{0.5}}{2}.\left(r^2 - x^2\right)^{0.5}.dx = \frac{1}{2}\left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$$

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MOMENTS OF INERTIA OF COMPOSITE AREAS

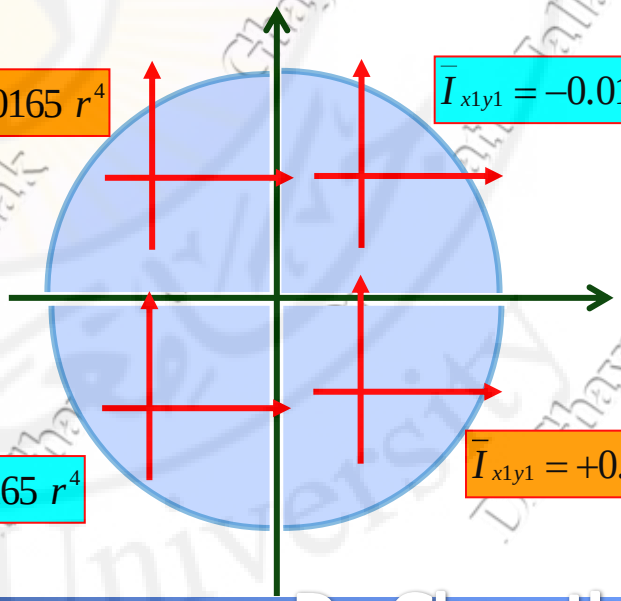
$$I_{xy} = \frac{r4}{8}$$



$$I_{XY} = \bar{I}_{x1y1} + (A_i) \times X_G \times Y_G \Rightarrow \bar{I}_{x1y1} = I_{XY} - (A_i) \times X_G \times Y_G$$

$$= \frac{r^4}{8} - \frac{\pi r^2}{4} \times \left(-\frac{4r}{3\pi}\right) \times \left(-\frac{4r}{3\pi}\right)$$

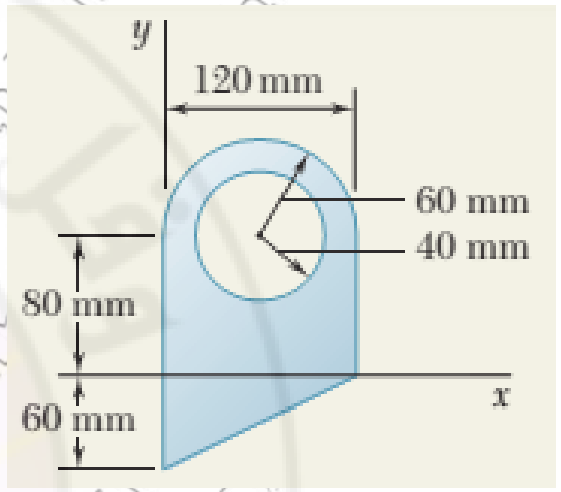
$$\bar{I}_{x1y1} = r^4\left(\frac{1}{8} - \frac{4}{9\pi}\right) = -0.0165\ r^4$$

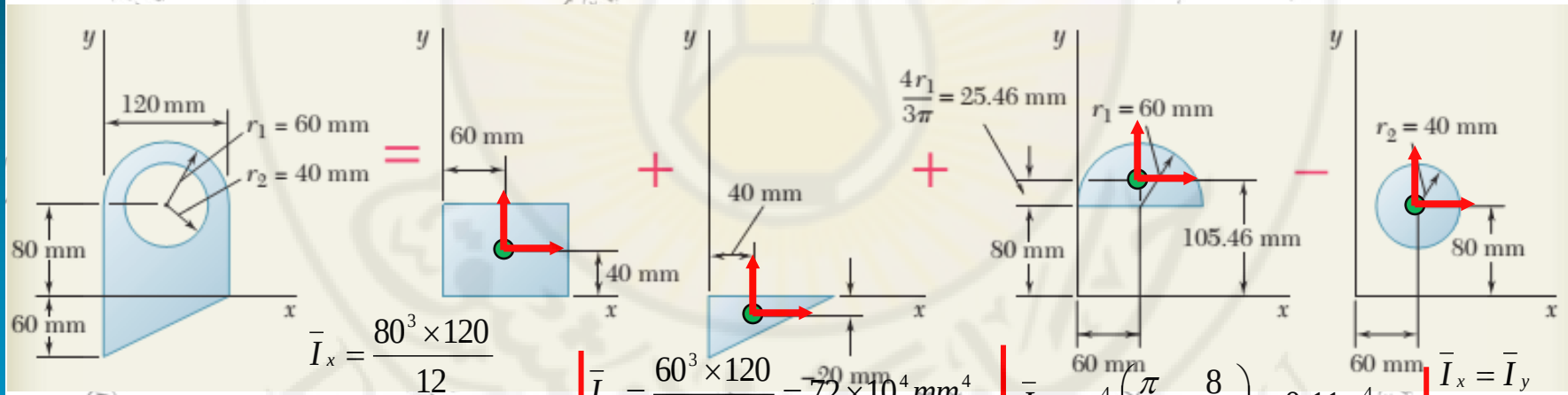


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MOMENTS OF INERTIA OF COMPOSITE AREAS

For the plane area shown, determine
(*a*) the location of the centroid.
(*b*) The moment of inertia about x & y axis.
(*C*) The moment of inertia about the centroidal axis.





$$\bar{I}_x = \frac{80^3 \times 120}{12}$$

$$\bar{I}_x = 512 \times 10^4\, mm^4$$

$$\bar{I}_y = \frac{120^3 \times 80}{12}$$

$$\bar{I}_y = 1152 \times 10^4\, mm^4$$

$$\bar{I}_x = \frac{60^3 \times 120}{36} = 72 \times 10^4\, mm^4$$

$$\bar{I}_y = \frac{120^3 \times 60}{36} = 288 \times 10^4\, mm^4$$

$$\bar{I}_{xy} = \frac{120^2 \times 60^2}{72} = 72 \times 10^4\, mm^4$$

$$\bar{I}_x = r^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) \approx 0.11\, r^4$$

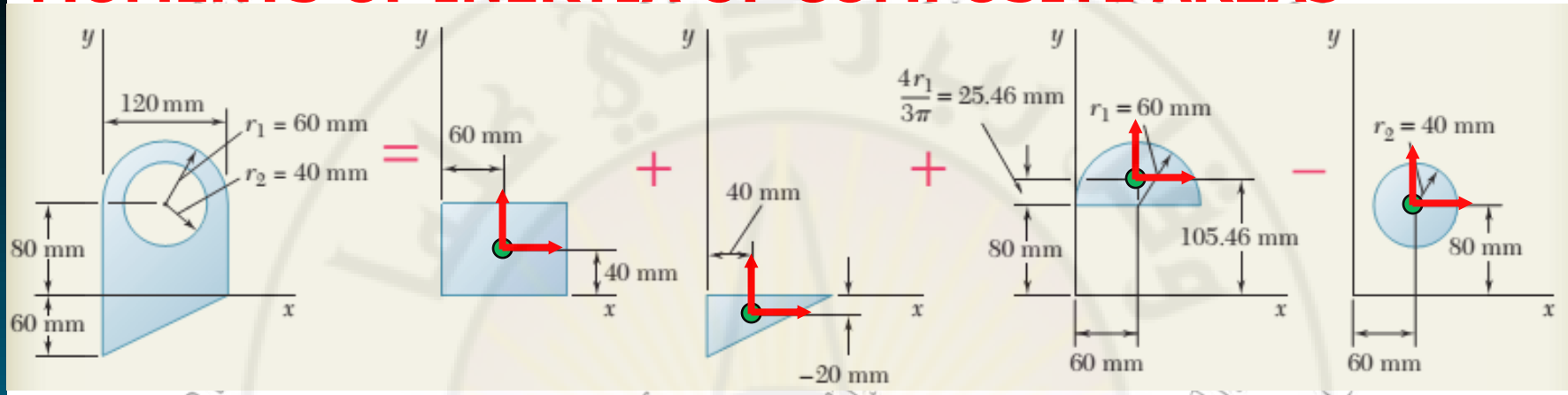
$$= 142.56 \times 10^4\, mm^4$$

$$\bar{I}_y = \frac{\pi \times r^4}{8} = 508.68 \times 10^4\, mm^4$$

$$\bar{I}_x = \bar{I}_y = \frac{\pi \times r^4}{4} = 200.96 \times 10^4\, mm^4$$

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MOMENTS OF INERTIA OF COMPOSITE AREAS

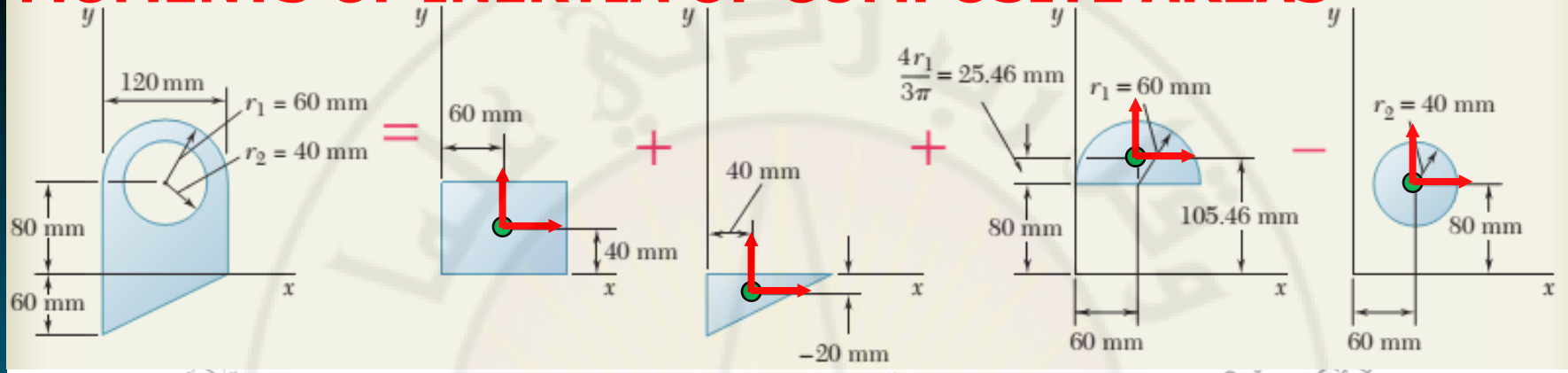


Seg	A_i mm^2	X_i mm	Y_i mm	$A_i \cdot x_i$ $x10^3$ mm^3	$A_i \cdot y_i$ $X10^3$ mm^3	$\bar{I}_x$ $x10^4$ mm^4	$\bar{I}_y$ $x10^4$ mm^4	$A_i.x_i^2$ $x10^4$ mm^4	$A_i.y_i^2$ $x10^4$ mm^4	$\bar{I}_{xy}$ $X10^4$ mm^4	$A_i.x_i.y_i$ $x10^4$ mm^4
Rect	9600	60	40	576	384	512	1152	3456	1536	0	2304
triang	3600	40	-20	144	-72	72	288	576	144	72	-288
Semi-cir	5652	60	105.5	339.1	596.3	142.6	508.7	2034.7	6290.8	0	3577.7
circle	-5024	60	80	-301.4	-401.9	-200.9	-200.9	-1808.6	-3215.4	0	-2411.5
Σ	13828			757.7	506.4	525.7	1747.8	4258.1	4755.4	72	3182.2

$$X_G = \frac{\sum A_i \times x_i}{\sum A_i} = \frac{757.7 \times 10^3}{13828} = 54.8mm \quad , \quad Y_G = \frac{\sum A_i \times y_i}{\sum A_i} = \frac{506.4 \times 10^3}{13828} = 36.6mm$$

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MOMENTS OF INERTIA OF COMPOSITE AREAS



Seg	A_i mm^2	X_i mm	Y_i mm	$A_i \cdot x_i$ $x10^3$ mm^3	$A_i \cdot y_i$ $X10^3$ mm^3	$\bar{I}_x$ $x10^4$ mm^4	$\bar{I}_y$ $x10^4$ mm^4	$A_i.x_i^2$ $x10^4$ mm^4	$A_i.y_i^2$ $x10^4$ mm^4	$\bar{I}_{xy}$ $X10^4$ mm^4	$A_i.x_i.y_i$ $x10^4$ mm^4
Rect	9600	60	40	576	384	512	1152	3456	1536	0	2304
triang	3600	40	-20	144	-72	72	288	576	144	72	-288
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circle	-5024	60	80	-301.4	-401.9	-200.9	-200.9	-1808.6	-3215.4	0	-2411.5
∑	13828			757.7	506.4	525.7	1747.8	4258.1	4755.4	72	3182.2

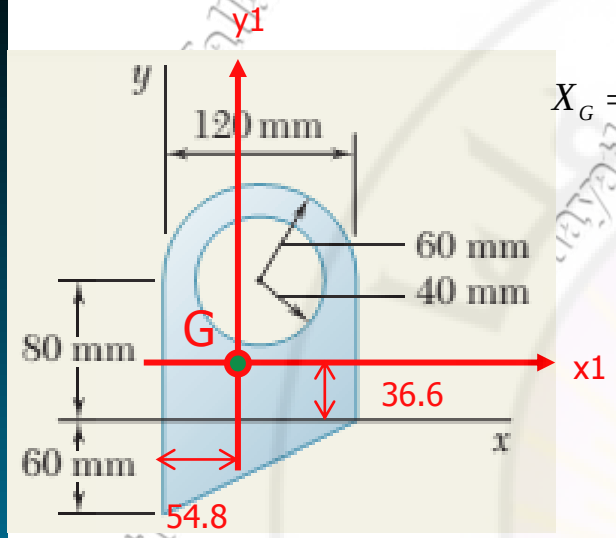
$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 525.7 \times 10^4 + 4755.4 \times 10^4 = 5281.1 \times 10^4 \, mm^4$$

$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 1747.8 \times 10^4 + 4258.1 \times 10^4 = 6005.9 \times 10^4 \, mm^4 \,,$$

$$I_{XY} = \sum \bar{I}_{xy} + \sum A_i \times x_i \times y_i = 72 \times 10^4 + 3182.2 \times 10^4 = 3254.2 \times 10^4 \, mm^4$$

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MOMENTS OF INERTIA OF COMPOSITE AREAS



$$X_G = \frac{\sum A_i \times x_i}{\sum A_i} = \frac{757.7\times10^3}{13828} = 54.8mm \quad , Y_G = \frac{\sum A_i \times y_i}{\sum A_i} = \frac{506.4\times10^3}{13828} = 36.6mm$$

$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 525.7\times10^4 + 4755.4\times10^4 = 5281.1\times10^4\, mm^4$$

$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 1747.8\times10^4 + 4258.1\times10^4 = 6005.9\times10^4\, mm^4 \ ,$$

$$I_{XY} = \sum \bar{I}_{xy} + \sum A_i \times x_i \times y_i = 72\times10^4 + 3182.2\times10^4 = 3254.2\times10^4\, mm^4$$

$$I_X = \bar{I}_{x1} + (\sum A_i)\times Y_G^2 \Rightarrow \bar{I}_{x1} = I_X - (\sum A_i)\times Y_G^2 = 5281.1\times10^4 - (13828)\times(-36.6)^2 = 3428.75\times10^4\, mm^4$$

$$I_Y = \bar{I}_{y1} + (\sum A_i)\times X_G^2 \Rightarrow \bar{I}_{y1} = I_Y - (\sum A_i)\times X_G^2 = 6005.9\times10^4 - (13828)\times(-54.8)^2 = 1853.3\times10^4\, mm^4$$

$$I_{XY} = \bar{I}_{x1y1} + (\sum A_i)\times X_G \times Y_G \Rightarrow \bar{I}_{x1y1} = I_{XY} - (\sum A_i)\times X_G \times Y_G = 3254.2\times10^4 - (13828)\times(-36.6)\times(-54.8)$$

$$\bar{I}_{x1y1} = 480.75\times10^4\, mm^4$$

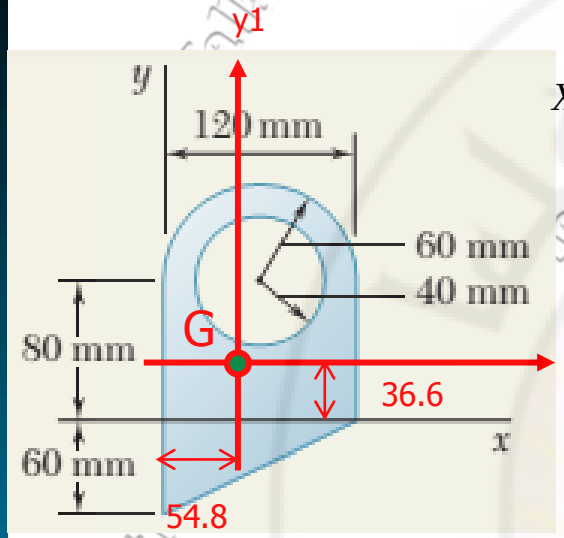
المحاور الجديدة

المحاور القديمة المركزية

البعد بين المحاور المتوازية منسوبة للمحاور القديمة المركزية

Dr. Ghayath Hallak

MOMENTS OF INERTIA OF COMPOSITE AREAS



$$X_G = \frac{\sum A_i \times x_i}{\sum A_i} = \frac{757.7\times10^3}{13828} = 54.8mm \quad , Y_G = \frac{\sum A_i \times y_i}{\sum A_i} = \frac{506.4\times10^3}{13828} = 36.6mm$$

$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 525.7\times10^4 + 4755.4\times10^4 = 5281.1\times10^4\, mm^4$$

$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 1747.8\times10^4 + 4258.1\times10^4 = 6005.9\times10^4\, mm^4 \,,$$

$$I_{XY} = \sum \bar{I}_{xy} + \sum A_i \times x_i \times y_i = 72\times10^4 + 3182.2\times10^4 = 3254.2\times10^4\, mm^4$$

$$I_X = \bar{I}_{x1} + (\sum A_i)\times Y_G^2 \Rightarrow \bar{I}_{x1} = I_X - (\sum A_i)\times Y_G^2 = 5281.1\times10^4 - (13828)\times(-36.6)^2 = 3428.75\times10^4\, mm^4$$

$$I_Y = \bar{I}_{y1} + (\sum A_i)\times X_G^2 \Rightarrow I_{y1} = I_Y - (\sum A_i)\times X_G^2 = 6005.9\times10^4 - (13828)\times(-54.8)^2 = 1853.3\times10^4\, mm^4$$

$$I_{XY} = \bar{I}_{x1y1} + (\sum A_i)\times X_G \times Y_G \Rightarrow \bar{I}_{x1y1} = I_{XY} - (\sum A_i)\times X_G \times Y_G = 3254.2\times10^4 - (13828)\times(-36.6)\times(-54.8)$$

$$\bar{I}_{x1y1} = 480.75\times10^4\, mm^4$$

$$tan2\phi_p = \frac{2\bar{I}_{x1y1}}{\bar{I}_{y1} - \bar{I}_{x1}} = \frac{2(480.75\times10^4)}{(1853.3-3428.75)\times10^4} = -0.61 \Rightarrow \phi_p = -15.7\ degree$$

$$I_{xp} = \bar{I}_{x1}\cos^2\phi_p + \bar{I}_{y1}\sin^2\phi_p - \bar{I}_{x1y1}\sin2\phi_p = 3563.8\times10^4\, mm^4$$

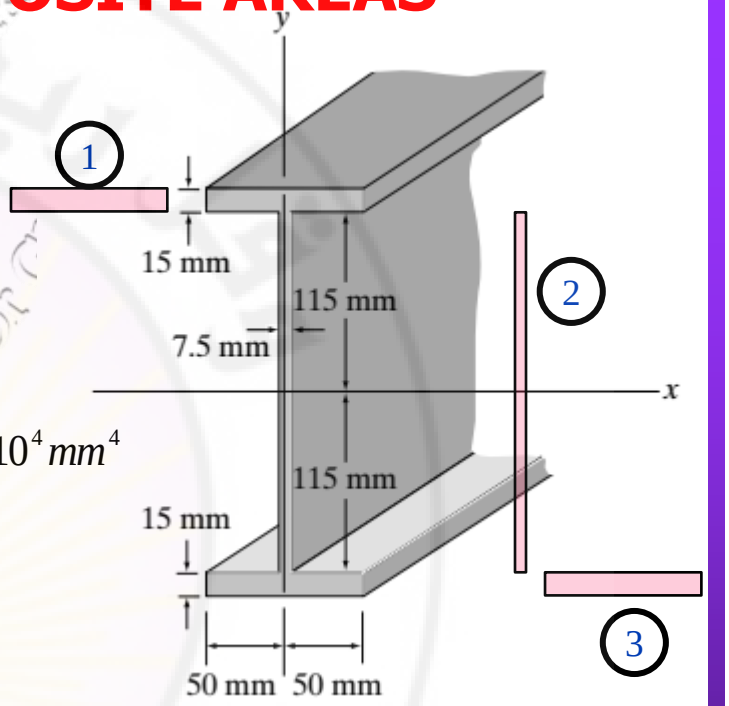
$$I_{yp} = \bar{I}_{y1}\cos^2\phi_p + \bar{I}_{x1}\sin^2\phi_p + \bar{I}_{x1y1}\sin2\phi_p = 1718.2\times10^4\, mm^4$$

$$I_{max,min} = \frac{\bar{I}_{x1} + \bar{I}_{y1}}{2} \pm \frac{1}{2}\sqrt{(\bar{I}_{x1} - \bar{I}_{y1})^2 + 4\bar{I}^2_{x1y1}} = \{3563.8\times10^4 \ , \ 1718.0\times10^4\}\, mm^4$$

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MOMENTS OF INERTIA OF COMPOSITE AREAS

Determine the moment of inertia of the beam's cross-sectional area about the x & y axis.



$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 766.07 \times 10^4 + 4501.8 \times 10^4 = 5267.87 \times 10^4 \, mm^4$$

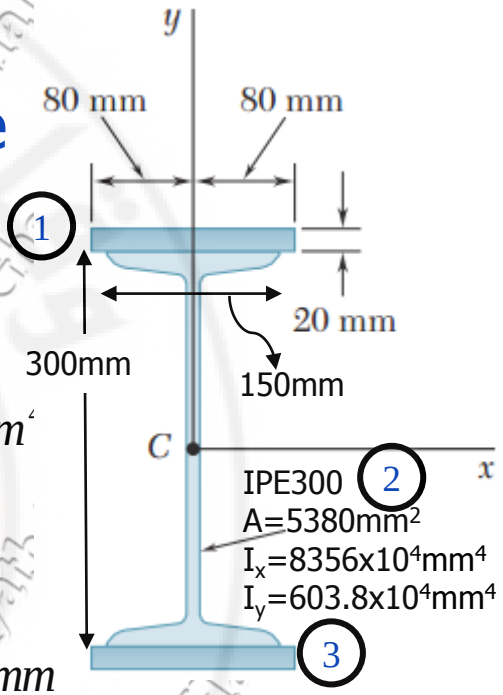
$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 250.809 \times 10^4 + 0 = 250.809 \times 10^4 \, mm^4 \, ,$$

Seg	A_i mm^2	X_i mm	Y_i mm	$A_i \cdot x_i$ $x10^3$ mm^3	$A_i \cdot y_i$ $X10^3$ mm^3	I_{xi} $x10^4$ mm^4	I_{yi} $x10^4$ mm^4	$A_i x_i^2$ $x10^4$ mm^4	$A_i y_i^2$ $x10^4$ mm^4
1	1500	0	122.5	0	183.75	2.8125	125	0	2250.9
2	1725	0	0	0	0	760.44	0.809	0	0
3	1500	0	-122.5	0	-183.75	2.8125	125	0	2250.9
Σ	4725			0	0	766.07	250.809	0	4501.8

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MOMENTS OF INERTIA OF COMPOSITE AREAS

Two 20-mm steel plates are welded to a rolled IPE 300 section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal *x* and *y* axes.



$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 8377.3\times10^4 + 16384\times10^4 = 9761.3\times10^4\, mm^4$$

$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 1969.14\times10^4 + 0 = 1969.14\times10^4\, mm^4$$

$$r_x = \sqrt{\frac{I_X}{A}} = \sqrt{\frac{9761.3\times10^4}{11780}} = 91mm, r_Y = \sqrt{\frac{I_Y}{A}} = \sqrt{\frac{1969.1\times10^4}{11780}} = 41mm$$

Seg	A_i mm^2	X_i mm	Y_i mm	$A_i \cdot x_i$ $x10^3$ mm^3	$A_i \cdot y_i$ $X10^3$ mm^3	I_{xi} $x10^4$ mm^4	I_{yi} $x10^4$ mm^4	$A_i.x_i^2$ $x10^4$ mm^4	$A_i.y_i^2$ $x10^4$ mm^4
1	3200	0	160	0	512	10.67	682.67	0	8192
2	5380	0	0	0	0	8356	603.8	0	0
3	3200	0	-160	0	-512	10.67	682.67	0	8192
Σ	11780			0	0	8377.3	1969.14	0	16384

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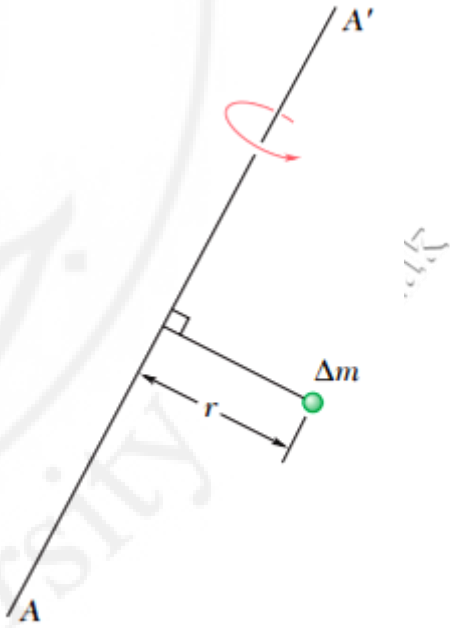
MOMENT OF INERTIA OF A MASS

Consider a small mass Δm mounted on a rod of negligible mass which can rotate freely about an axis AA'. If a couple is applied to the system, the rod and mass, assumed to be initially at rest, will start rotating about AA'.

The time required for the system to reach a given speed of rotation is proportional to the mass Δm and to the square of the distance r.

The product $r^2 \Delta m$ provides, therefore, a measure of the *inertia* of the system, i.e., a measure of the resistance the system offers when we try to set it in motion.

For this reason, the product $r^2 \Delta m$ is called the *moment of inertia* of the mass Δm with respect to the axis AA'.

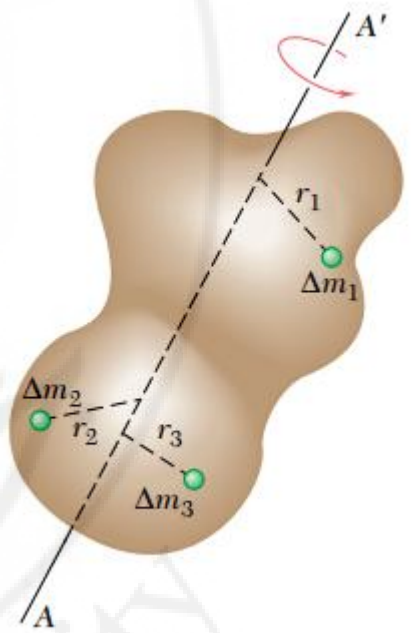


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1- MOMENT OF INERTIA OF A MASS

Consider now a body of mass m which is to be rotated about an axis AA'. Dividing the body into elements of mass $\Delta m1$, $\Delta m2$, etc., we find that the body's resistance to being rotated is measured by the sum $(r_1)^2 \Delta m_1 + (r_2)^2 \Delta m_2 + \ldots$

$$I_{AA'} = \int r^2 \, dm$$

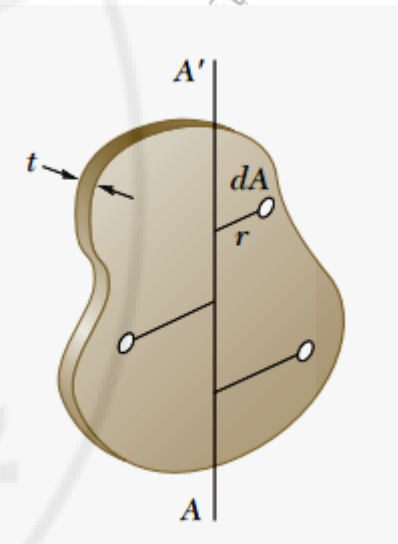


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MOMENTS OF INERTIA OF THIN PLATES

Consider a thin plate of uniform thickness t, which is made of a homogeneous material of density ρ (density = mass per unit volume). The mass moment of inertia of the plate with respect to an axis AA' *contained in the plane* of the plate

$$I_{AA',mass} = \int r^2 \, dm$$



Since $dm = \rho \, t \, dA$, we write

$$I_{AA',mass} = \rho t \int r^2 \, dA$$

But r represents the distance of the element of area dA to the axis AA'; the integral is therefore equal to the moment of inertia of the area of the plate with respect to AA'. We have

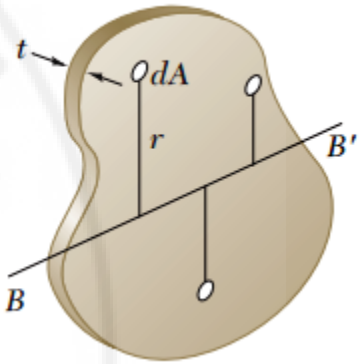
$$I_{AA',mass} = \rho t \, I_{AA',area}$$

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MOMENTS OF INERTIA OF THIN PLATES

Similarly, for an axis *BB'* which is contained in the plane of the plate and is perpendicular to *AA'*, we have

$$I_{BB',mass} = \rho t\, I_{BB',area}$$



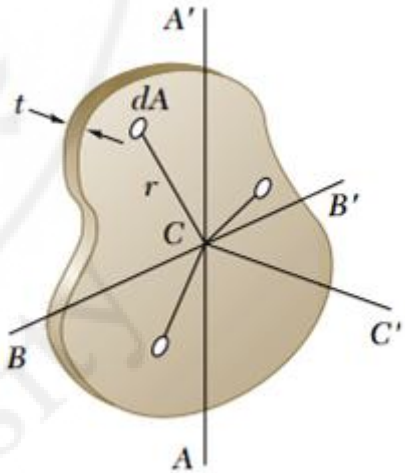
Considering now the axis *CC'* which is *perpendicular* to the plate and passes through the point of intersection *C* of *AA'* and *BB'*, we write

$$I_{CC',mass} = \rho t\, J_{C,area}$$

where J_C is the *polar* moment of inertia of the area of the plate with respect to point *C*.

$$J_C = I_{AA'} + I_{BB'}$$

$$I_{CC',mass} = I_{AA',mass} + I_{BB',mass}$$

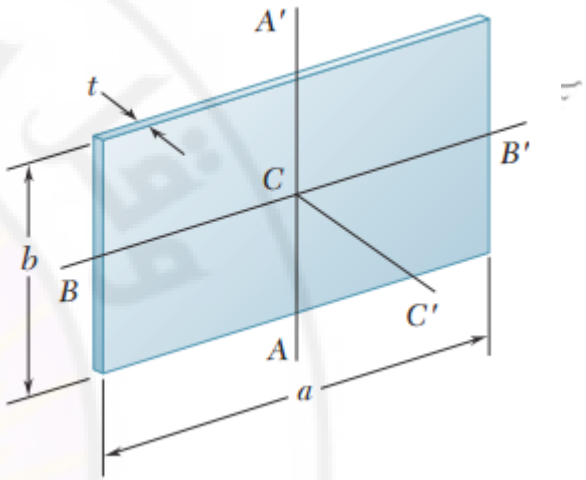


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MOMENTS OF INERTIA OF THIN PLATES

Rectangular Plate.

The mass moments of inertia with respect to axes through the center of gravity of the plate:



$$I_{AA',mass} = \rho t\, I_{AA',area} = \rho t\,(a^3 b/12)$$

$$I_{BB',mass} = \rho t\, I_{BB',area} = \rho t\,(b^3 a/12)$$

$$\rho\, abt = m$$

$$I_{AA',mass} = m\, a^2/12$$

$$I_{BB',mass} = m\, b^2/12$$

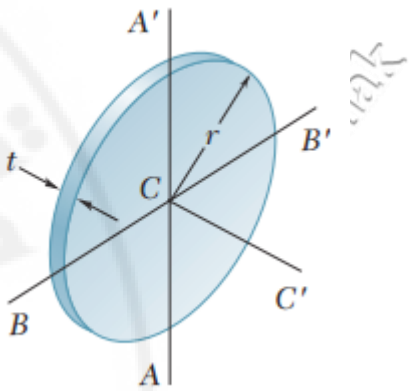
$$I_{CC',mass} = I_{AA',mass} + I_{BB',mass} = m\,[(a^2 + b^2)/12]$$

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MOMENTS OF INERTIA OF THIN PLATES

Circular Plate.

The mass moments of inertia with respect to axes through the center of gravity of the plate:



$$I_{AA',mass} = \rho t \, I_{AA',area} = \rho t \, (\pi \, r^4 / 4)$$

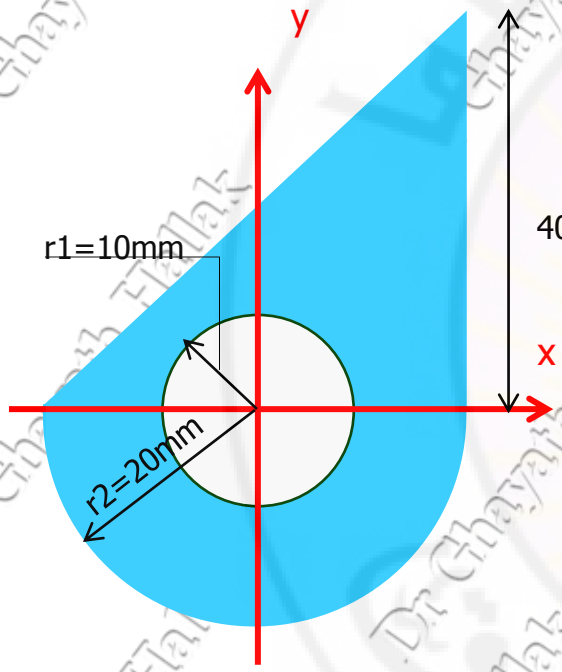
$$\rho \, \pi \, r^2 \, t = m$$

$$I_{AA',mass} = I_{BB',mass} = m \, r^2 / 4$$

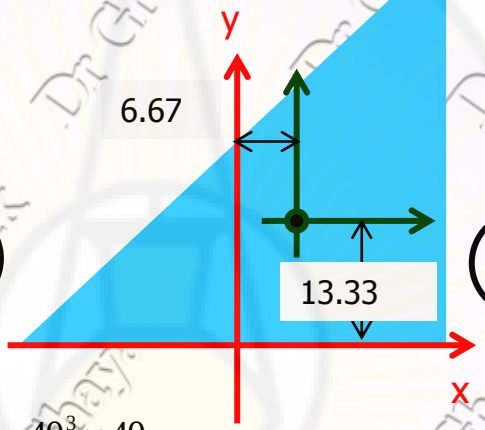
$$I_{CC',mass} = I_{AA',mass} + I_{BB',mass} = m \, r^2 / 2$$

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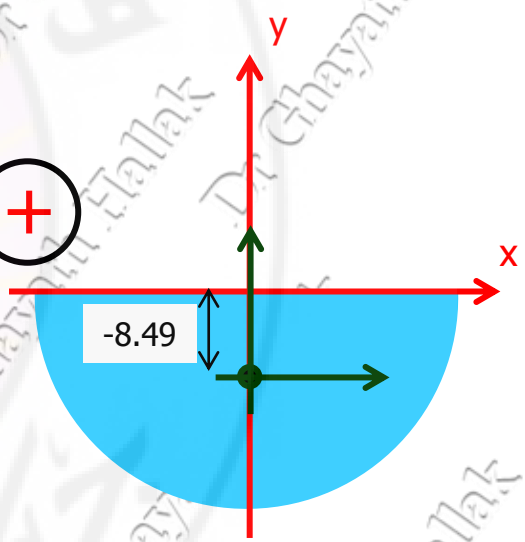
مثال : اوجد عزوم العطالة وجداء العطالة للسطح المبين وذلك وفقاً للمحاور المفروضة .وكذلك اوجد عزوم العطالة وجداء العطالة للمحاور المركزية.



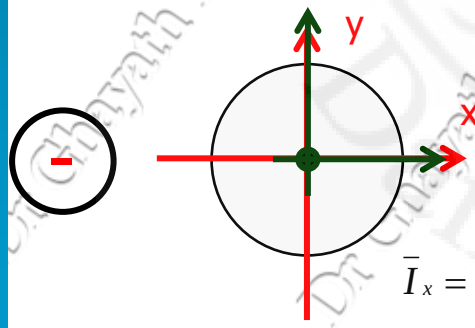
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+



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$$\bar{I}_x = \frac{40^3 \times 40}{36} = 7.1 \times 10^4 mm^4$$

$$\bar{I}_y = \frac{40^3 \times 40}{36} = 7.1 \times 10^4 mm^4$$

$$\bar{I}_{xy} = \frac{40^2 \times 40^2}{72} = 3.56 \times 10^4 mm^4$$

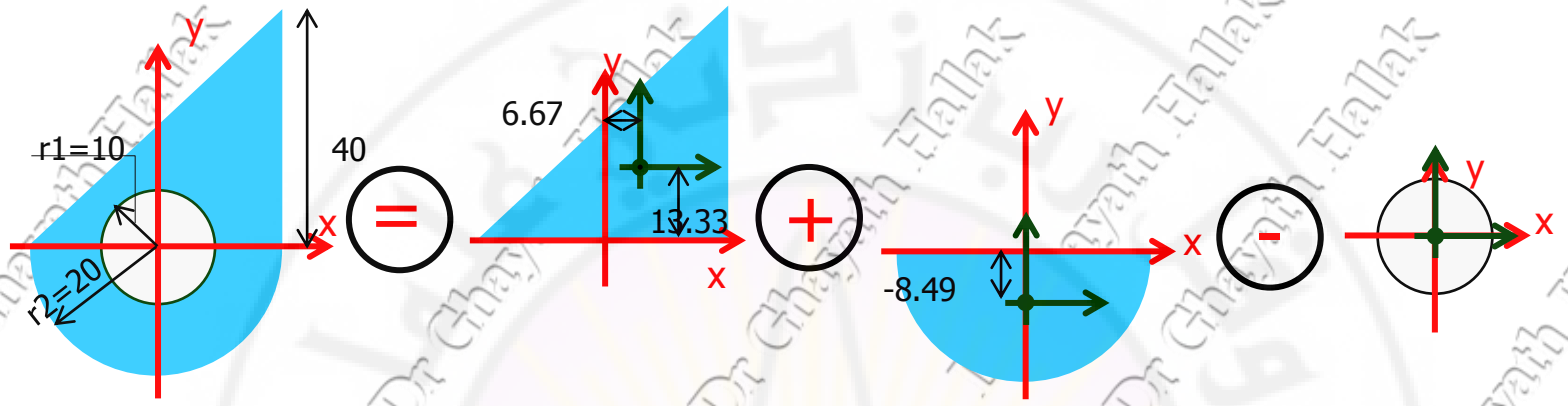
$$\bar{I}_x = r^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) \approx 0.11\, r^4$$

$$= 1.76 \times 10^4 mm^4$$

$$\bar{I}_y = \frac{\pi \times r^4}{8} = 6.28 \times 10^4 mm^4$$

$$\bar{I}_x = \bar{I}_y = \frac{\pi \times r^4}{4} = 0.785 \times 10^4 mm^4$$

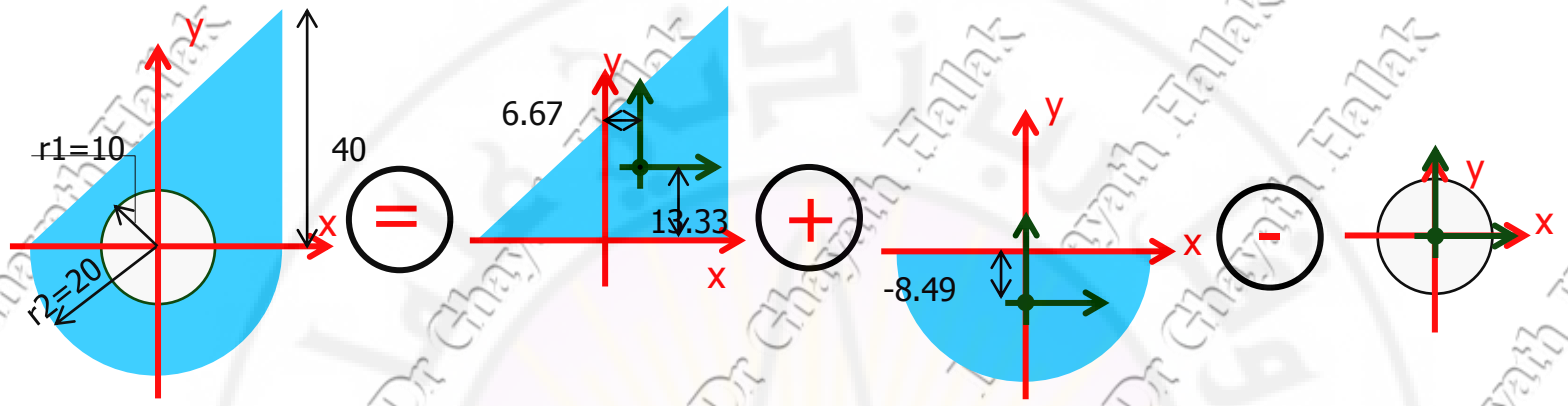
Dr. Ghayath Hallak



Seg	A_i mm^2	X_i mm	Y_i mm	$A_i \cdot x_i$ $x10^3$ mm^3	$A_i \cdot y_i$ $X10^3$ mm^3	$\bar{I}_x$ $x10^4$ mm^4	$\bar{I}_y$ $x10^4$ mm^4	$A_i.x_i^2$ $x10^4$ mm^4	$A_i.y_i^2$ $x10^4$ mm^4	$\bar{I}_{xy}$ mm^4	$A_i.x_i.y_i$ $x10^4$ mm^4
triang	800	6.67	13.33	5.336	10.66	7.1	7.1	3.56	14.22	3.56	7.113
Semi-cir	628	0	-8.49	0	-5.331	1.76	6.28	0	4.526	0	0
circle	-314	0	0	0	0	0.785	0.785	0	0	0	0
Σ	1111			5.336	5.329	8.075	12.595	3.56	18.746	3.56	7.113

$$X_G = \frac{\sum A_i \times x_i}{\sum A_i} = \frac{5.336 \times 10^3}{1111} = 4.8mm \quad , Y_G = \frac{\sum A_i \times y_i}{\sum A_i} = \frac{5.329 \times 10^3}{1111} = 4.8mm$$

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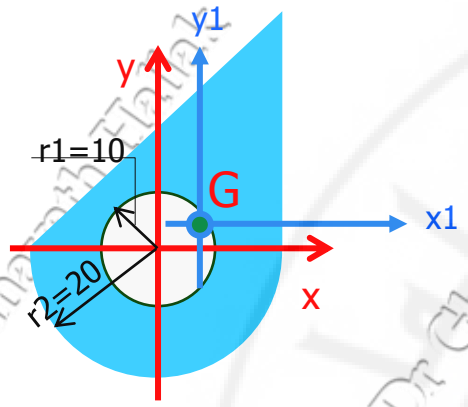
Seg	A_i mm^2	X_i mm	Y_i mm	$A_i \cdot x_i$ $x10^3$ mm^3	$A_i \cdot y_i$ $X10^3$ mm^3	$\bar{I}_x$ $x10^4$ mm^4	$\bar{I}_y$ $x10^4$ mm^4	$A_i.x_i^2$ $x10^4$ mm^4	$A_i.y_i^2$ $x10^4$ mm^4	$\bar{I}_{xy}$ mm^4	$A_i.x_i.y_i$ $x10^4$ mm^4
triang	800	6.67	13.33	5.336	10.66	7.1	7.1	3.56	14.22	3.56	7.113
Semi-cir	628	0	-8.49	0	-5.331	1.76	6.28	0	4.526	0	0
circle	-314	0	0	0	0	0.785	0.785	0	0	0	0
Σ	1111			5.336	5.329	8.075	12.595	3.56	18.746	3.56	7.113

$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 8.075 \times 10^4 + 18.746 \times 10^4 = 26.821 \times 10^4 \, mm^4$$

$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 12.595 \times 10^4 + 3.56 \times 10^4 = 16.155 \times 10^4 \, mm^4 \, ,$$

$$I_{XY} = \sum \bar{I}_{xy} + \sum A_i \times x_i \times y_i = 3.56 \times 10^4 + 7.113 \times 10^4 = 10.673 \times 10^4 \, mm^4$$

Dr. Ghayath Hallak



المحاور المركزية

$$I_X = \bar{I}_{x1} + (\sum A_i) \times Y_G^2 \Rightarrow \bar{I}_{x1} = I_X - (\sum A_i) \times Y_G^2 = 26.821 \times 10^4 - (1111) \times (-4.8)^2 = 24.26 \times 10^4 \, mm^4$$

$$I_Y = \bar{I}_{y1} + (\sum A_i) \times X_G^2 \Rightarrow \bar{I}_{y1} = I_Y - (\sum A_i) \times X_G^2 = 16.155 \times 10^4 - (1111) \times (-4.8)^2 = 13.595 \times 10^4 \, mm^4$$

$$I_{XY} = \bar{I}_{x1y1} + (\sum A_i) \times X_G \times Y_G \Rightarrow \bar{I}_{x1y1} = I_{XY} - (\sum A_i) \times X_G \times Y_G = 10.673 \times 10^4 - (1111) \times (-4.8) \times (-4.8)$$

$$\bar{I}_{x1y1} = 8.11 \times 10^4 \, mm^4$$

المحاور الرئيسية

$$tan 2\phi_p = \frac{2\bar{I}_{x1y1}}{\bar{I}_{y1} - \bar{I}_{x1}} = \frac{2(8.11 \times 10^4)}{(13.595 - 24.26) \times 10^4} = -1.52 \Rightarrow \phi_p = -28.34 \, degree$$

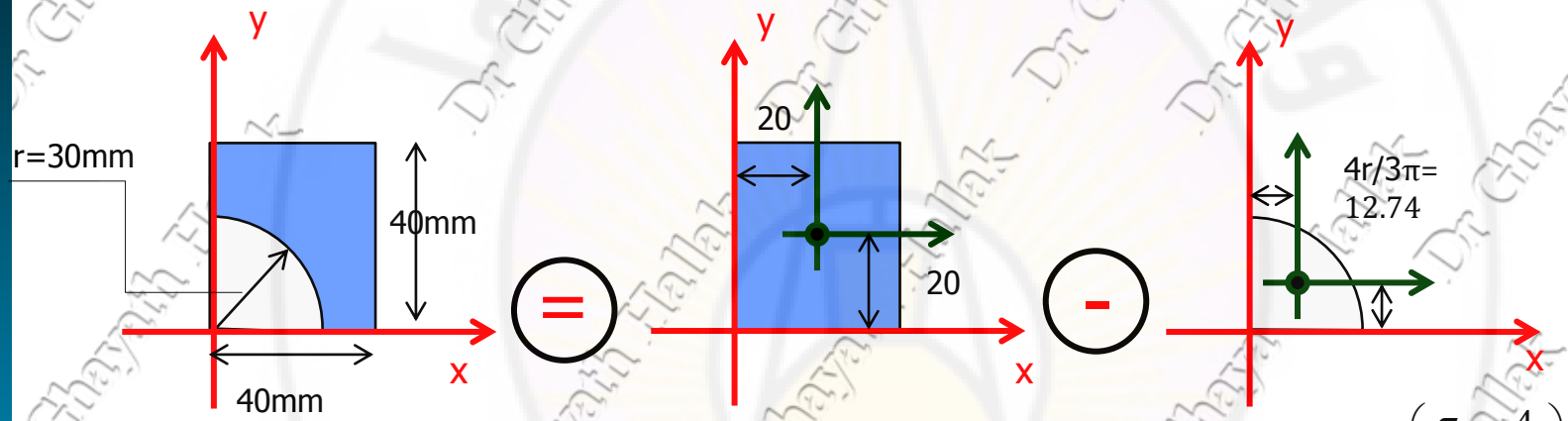
$$I_{xp} = \bar{I}_{x1} cos^2 \phi_p + \bar{I}_{y1} sin^2 \phi_p - \bar{I}_{x1y1} sin 2\phi_p = 28.634 \times 10^4 \, mm^4$$

$$I_{yp} = \bar{I}_{y1} cos^2 \phi_p + \bar{I}_{x1} sin^2 \phi_p + \bar{I}_{x1y1} sin 2\phi_p = 9.221 \times 10^4 \, mm^4$$

$$I_{max,min} = \frac{\bar{I}_{x1} + \bar{I}_{y1}}{2} \pm \frac{1}{2} \sqrt{(\bar{I}_{x1} - \bar{I}_{y1})^2 + 4\bar{I}^2_{x1y1}} = \{28.634 \times 10^4 \ , \ 9.221 \times 10^4\} mm^4$$

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مثال : اوجد عزوم العطالة وجداء العطالة للسطح المبين وذلك وفقاً للمحاور المفروضة .وكذلك اوجد عزوم العطالة وجداء العطالة للمحاور المركزية .



$$\bar{I}_x = \frac{40^3 \times 40}{12} = 21.33 \times 10^4 mm^4$$

$$\bar{I}_y = \frac{40^3 \times 40}{12} = 21.33 \times 10^4 mm^4$$

$$\bar{I}_{xy} = 0 mm^4$$

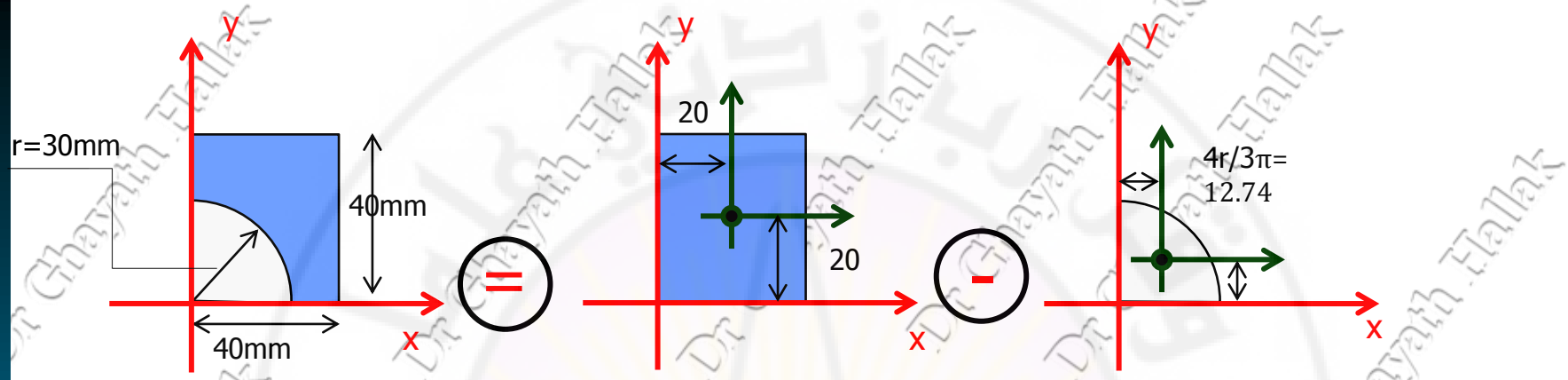
$$\bar{I}_x = \bar{I}_y = r^4 \left(\frac{\pi}{16} - \frac{4}{9\pi} \right) \approx 0.055\, r^4$$

$$= 4.455 \times 10^4 mm^4$$

$$\bar{I}_{xy} = r^4 \left(\frac{1}{8} - \frac{4}{9\pi} \right) = -0.0165\, r^4$$

$$\bar{I}_{xy} = -1.337 \times 10^4 mm^4$$

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Seg	A_i mm²	X_i mm	Y_i mm	$A_i \cdot x_i$ x10³ mm³	$A_i \cdot y_i$ X10³ mm³	$\bar{I}_x$ x10⁴ mm⁴	$\bar{I}_y$ x10⁴ mm⁴	$A_i.x_i^2$ x10⁴ mm⁴	$A_i.y_i^2$ x10⁴ mm⁴	$\bar{I}_{xy}$ mm⁴	$A_i.x_i.y_i$ x10⁴ mm⁴
Squar	1600	20	20	32	32	21.33	21.33	64	64	0	64
Quart er-cir	706.5	12.74	12.74	9	9	4.455	4.455	11.47	11.47	-1.34	11.47
∑	893.5			23	23	16.875	16.875	52.53	52.53	+1.34	52.53

$$X_G = \frac{\sum A_i \times x_i}{\sum A_i} = \frac{23\times10^3}{893.5} = 25.7mm \quad , Y_G = \frac{\sum A_i \times y_i}{\sum A_i} = \frac{23\times10^3}{893.5} = 25.7mm$$

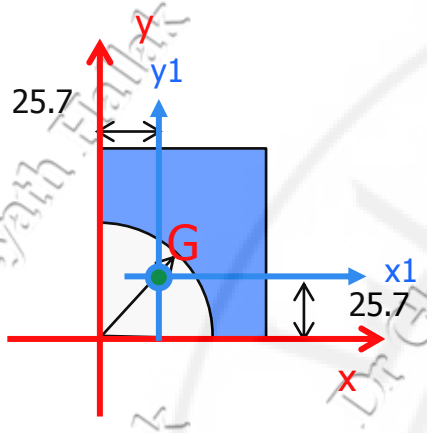
$$I_X = \sum \bar{I}_x + \sum A_i \times y_i^2 = 16.875\times10^4 + 52.53\times10^4 = 69.405\times10^4 mm^4$$

$$I_Y = \sum \bar{I}_y + \sum A_i \times x_i^2 = 16.875\times10^4 + 52.53\times10^4 = 69.405\times10^4 mm^4 \ ,$$

$$I_{XY} = \sum \bar{I}_{xy} + \sum A_i \times x_i \times y_i = 1.34\times10^4 + 52.53\times10^4 = 53.87\times10^4 mm^4$$

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المحاور المركزية



$$I_X = \bar{I}_{x1} + (\sum A_i) \times Y_G^2 \Rightarrow \bar{I}_{x1} = I_X - (\sum A_i) \times Y_G^2 = 69.405 \times 10^4 - (893.5) \times (-25.7)^2 = 10.39 \times 10^4 \, mm^4$$

$$I_Y = \bar{I}_{y1} + (\sum A_i) \times X_G^2 \Rightarrow \bar{I}_{y1} = I_Y - (\sum A_i) \times X_G^2 = 69.405 \times 10^4 - (893.5) \times (-25.7)^2 = 10.39 \times 10^4 \, mm^4$$

$$I_{XY} = \bar{I}_{x1y1} + (\sum A_i) \times X_G \times Y_G \Rightarrow \bar{I}_{x1y1} = I_{XY} - (\sum A_i) \times X_G \times Y_G = 53.87 \times 10^4 - (893.5) \times (-25.7) \times (-25.7)$$

$$\bar{I}_{x1y1} = -5.145 \times 10^4 \, mm^4$$

المحاور الرئيسية

$$tan 2\phi_p = \frac{2\bar{I}_{x1y1}}{\bar{I}_{y1} - \bar{I}_{x1}} = \frac{2(-5.145 \times 10^4)}{(10.39 - 10.39) \times 10^4} = -\infty \Rightarrow \phi_p = -45 \, degree$$

$$I_{xp} = \bar{I}_{x1} \cos^2 \phi_p + \bar{I}_{y1} \sin^2 \phi_p - \bar{I}_{x1y1} \sin 2\phi_p = 5.245 \times 10^4 \, mm^4$$

$$I_{yp} = \bar{I}_{y1} \cos^2 \phi_p + \bar{I}_{x1} \sin^2 \phi_p + \bar{I}_{x1y1} \sin 2\phi_p = 15.535 \times 10^4 \, mm^4$$

$$I_{max,min} = \frac{\bar{I}_{x1} + \bar{I}_{y1}}{2} \pm \frac{1}{2} \sqrt{(\bar{I}_{x1} - \bar{I}_{y1})^2 + 4\bar{I}^2_{x1y1}} = \{15.535 \times 10^4 \ , \ 5.245 \times 10^4\} mm^4$$

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